ZIGZAG DIAGRAMS AND MARTIN BOUNDARY

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Abstract. In this paper, we investigate the Martin boundary of a graded graph $\mathcal{Z}$ whose paths represent coherent sequences of permutations. This graph was introduced by Viennot and its boundary studied by Gnedin and Olshanski. We prove that the Martin boundary of this graph coincides with its minimal boundary. Then we relate paths on this graph with paths on the Young graph, and finally give a central limit theorem for the Plancherel measure on the paths of $\mathcal{Z}$.

1. Introduction

The lattice $\mathcal{Z}$ of zigzag diagrams is a graded graph whose vertices of degree n are labelled by compositions of n (which can be seen as ribbon Young diagrams). The study of this kind of lattices drew increasing interests these last decades, due to their interactions with representations of semi-simple algebras and with discrete random walks. In particular an other example of graded graph, the Young lattice $\mathcal{Y}$, has been deeply studied by Vershik, Kerov and other authors (see [11] for a review on the subject), yielding major breakthroughs on the representation theory of S_∞ and on the asymptotic study of certain particle systems. As explained in [8], the lattices $\mathcal{Z}$ and $\mathcal{Y}$ are somehow related, since the latter can be seen as a projection of the former.

The connection between the lattice structure and its probabilistic applications is made through the study of harmonic functions on the associated graph. One of the first tasks is therefore the characterization of harmonic functions on the lattice ; it is then possible to single out particular harmonic functions and study the random variables they generate. A general framework for the representation of harmonic functions on a graph E was initiated by Martin in [15] with the concept of Martin boundary $\partial_M E$ and minimal boundary $\partial E_{\min}$: the Martin boundary is a topological space associated to the graph that allows to establish a bijection between positive harmonic functions and measures whose support is included in a particular subset of $\partial_M E$. The latter subset is precisely the minimal boundary $\partial E_{\min}$. It is therefore important to know both $\partial_M E$ and $\partial E_{\min}$ to provide a topological and measure theoretic approach to the kernel representation of harmonic functions (see [4] for an exhaustive review on the subject).

In general $\partial E_{\min}$ is strictly included in $\partial_M E$. However in many cases the two coincide, as it happens for example for the lattice $\mathcal{Y}$. In this paper we prove that the two boundaries also coincide for the lattice $\mathcal{Z}$. The minimal boundary of $\mathcal{Z}$ has

already been described by Gnedin and Olshanski in[8], through the so-called oriented paintbox construction, and thus it remains to prove that any element of the Martin boundary fits in this construction. As an application we provide a precise link between harmonic measures on $\mathcal{Y}$ and harmonic measures on $\mathcal{Z}$: this link was already exposed in [8], and in the present article we explain this relation by mapping directly paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$. Finally we study the behavior of a random path with respect to the Plancherel measure by providing a Central Limit Theorem.

Section 2 and 3 are devoted to preliminaries : the first gives necessary backgounds on Martin boundary, and the second describes the graph $\mathcal{Z}$ together with its link with compositions. The results of Gnedin and Olshanski on this graph are given in Section 4. In this section we provide also the pattern of the proof for the identification of the Martin boundary.

The proof heavily relies on combinatorics of compositions. In particular the Martin kernel of $\mathcal{Z}$, a two parameters function that characterizes the Martin boundary, is related to standard fillings of compositions. Two constructions are needed in order to identify the Martin boundary: Section 5 deals with the first one, which is the construction of a sequence of random variables that relates the Martin kernel to the oriented Paintbox construction of Gnedin and Olshanski. The second one has been done in [19] and is a general framework that gives combinatoric estimates on compositions. Some results of [19] are recalled in Section 6. Section 7 and 8 show the identification of the Martin boundary. Finally Section 9 gives the map between paths on $\mathcal{Z}$ and paths on $\mathcal{Y}$ and exposes probabilistic results associated to a particular point of the Martin boundary, called the Plancherel measure (due to its relations with the Plancherel measure on the graph $\mathcal{Y}$).

We should stress that the map between the paths on the two graphs appears clearly by using the algebra **FQSym** of Free Quasi-Symmetric functions; although this algebra won't be explained in this paper, the interested reader should refer to the Chapter 3 of [5] for an introduction to **FQSym** and an explanation of the construction we are doing in Section 9.2 of the present paper.

2. Graded graphs and Martin boundary

This section is a discussion that introduces the concept of Martin boundary and motivates its role in the framework of graded graphs. All these results and proofs can be found in [4].

2.1. **Graded graphs and random walk.**

The notations used here are from [18]. A rooted graded graph $\mathcal{G}$ is the data of a triple (V, ρ, E) where :

- V is a denumerable set of vertices with a distinguished element μ_0.
- $\rho : V \to \mathbb{N}$ is a rank function with $\rho^{-1}(\{0\}) = \{\mu_0\}$.
- The adjacency matrix E is a $V \times V$-matrix with entries in $\{0, 1\}$, such that $E(\mu, \nu)$ is zero if $\rho(\nu) \neq \rho(\mu) + 1$.

We write $\mu \nearrow \nu$ if $E(\mu,\nu)=1$. A path on $\mathcal{G}$ is sequence of vertices $(\mu_1,\dots,\mu_n,\dots)$ of increasing degree such that for all $i \geq 1$, $\mu_i \nearrow \mu_{i+1}$. For a given graded graph the paths counting function $d:\mathcal{V}\to\mathbb{N}^*$ is the function that gives for each vertex $\mu\in\mathcal{G}$ the number of paths between μ_0 and μ. When the endpoints of a path are not specified, the path is considered as an infinite path starting at the root.
There is a natural way of constructing random walks that respect the structure of the graph $\mathcal{G}$: such a random walk starts at μ_0, and at each step the successor is chosen according to a transition matrix P, with the condition that $P(\mu,\nu)=0$ if $E(\mu,\nu)=0$ and $\sum_\nu P(\mu,\nu)=1$.
Thus each transition matrix P associates to any path $\lambda=(\mu_0\nearrow\mu_1\cdots\nearrow\mu_n)$ a weight p_λ which is the probability of the realization of λ, namely

$$p_\lambda=\mathbb{P}(X_0=\mu_0,X_1=\mu_1,\dots,X_n=\mu_n)=\frac{1}{Z}P(\mu_0,\mu_1)\dots P(\mu_{n-1},\mu_n).$$

For some transition matrices P on $\mathcal{G}$, the weight $p(\lambda)$ only depends on the final vertex of the path (in this case we write $p(\lambda)=p(\mu)$ for any path λ between μ_0 and μ); such a transition matrix is called an harmonic matrix. In this case, a staightforward computation shows that p, the associated weight function, must verify

$$p(\mu)=\sum_{\mu\nearrow\nu}p(\nu), \tag{1}$$

and conversely, any positive solution p of (1) such that $p(\mu_0)=1$ yields an harmonic matrix. This can be interpreted in terms of potential theory.
Let X be a denumerable states space with transition matrix P. Let $H(X,P)^+$ (resp. $M(X,P)$) denote the set of positive harmonic functions (resp. positive harmonic measures), which is the set of functions $f:X\to\mathbb{R}^+$ satisfying $\sum_y P(x,y)f(y)=f(x)$ (resp. $\sum_x f(x)P(x,y)=f(y)$). For each $\alpha\in M(X,P)$ let the dual transition matrix P^t_α be defined by the expression

$$P^t_\alpha(x,y)=\mathbf{1}_{\alpha(x)\neq 0}\frac{\alpha(y)}{\alpha(x)}P(y,x),$$

if $x\neq y$, and $P^t(x,x)=1-\sum_{x\neq y}P^t(x,y)$. Then P^t_α is indeed a transition matrix on X and the following maps are well-defined:

$$H_\alpha:\begin{cases} H(X,P)^+ & \to & M(X,P^t_\alpha)\\ h & \mapsto & (x\mapsto \mathbf{1}_{\alpha(x)>0}\frac{1}{\alpha(x)}h(x))\end{cases},$$

and

$$M_\alpha:\begin{cases} M(X,P) & \to & H(X,P^t_\alpha)^+\\ m & \mapsto & (x\mapsto \alpha(x)m(x))\end{cases}.$$

The two maps are bijective if $\alpha>0$ on X.
Let P be a transition matrix on a graded graph $\mathcal{G}$; by a recursive computation there exists a unique invariant measure α_P with respect to P such that $\alpha_P(\mu_0)=1$. If P_p is

an harmonic matrix, with p the associated weight function, then $P(\mu,\nu)=\mathbf{1}_{\mu\nearrow\nu}\frac{p(\nu)}{p(\mu)}$ and $\alpha_p=d(\mu)p(\mu)$. Thus the dual transition matrix is

$$P^t_{\alpha_p}(\nu,\mu)=\mathbf{1}_{\mu\nearrow\nu}\frac{d(\mu)p(\mu)}{d(\nu)p(\nu)}\frac{p(\nu)}{p(\mu)}=\mathbf{1}_{\mu\nearrow\nu}\frac{d(\mu)}{d(\nu)}.$$

In particular $P^t_{\alpha_p}$ is independent of p and, by H_{α_p}, any harmonic function of P comes from an invariant measure of P^t. Conversely let α be an invariant measure of P^t. Then the dual matrix $(P^t)^t_\alpha$ is exactly $P_{\alpha/d}$, the harmonic matrix associated to the weight function $p=\alpha/d$. We can check that the duality yields indeed a bijection between harmonic matrices of $\mathcal{G}$ and elements of $M(\mathcal{G},P^t)$ taking the value 1 on μ_0. Thus the problem of finding the harmonic matrices on $\mathcal{G}$ is equivalent to the dual problem of finding harmonic measures with respect to P^t. Moreover an answer to the latter problem gives also by duality all the harmonic functions with respect to an harmonic matrix.
Fortunately a general framework, the Martin entrance boundary, describes exactly the harmonic measures associated to a transition matrix.

2.2. **Martin entrance boundary.** Let us take a closer look at the Markov chain $(\mathcal{G},P^t)$. Let $n_0\geq 1$ and ν a vertex of degree n_0. The random walk $X=(X_n)_{n\geq 0}$ with transition matrix P^t and initial distribution δ_ν goes backward from ν to μ_0 and stops at μ_0 at the times n_0. Let λ be a path between μ_0 and ν; from the definition of the kernel P^t, the probability that X follows the path λ is independent of λ and is therefore $\frac{1}{d(\nu)}$.
For μ of degree $m\leq n_0$, denote by $d(\mu,\nu)$ the number of paths between μ and ν (and by extension $d(\mu,\nu)=0$ if the degree of μ is larger than the one of ν). By counting the paths going from μ_0 to ν and passing through μ, the probability that $X_{n_0-m}=\mu$ is thus

$$\mathbb{P}(X_{n_0-m}=\mu)=\frac{d(\mu)d(\mu,\nu)}{d(\nu)}.$$

In particular setting $\alpha_\nu(\mu)=\frac{d(\mu)d(\mu,\nu)}{d(\nu)}$ yields a measure α_ν that is harmonic with respect to P^t, except on the vertex ν. To construct actual harmonic measures, it seems thus natural to look at the behavior of α_ν when $\nu\to\infty$. Making the latter rigorous requires to specify a convergence mode for sequences of vertices of increasing degree. Let $K_\mu(\nu)=\frac{d(\mu,\nu)}{d(\nu)}$ be the Martin kernel of $\mathcal{G}$, and define on $\mathcal{G}$ the metric :

$$D(\nu_1,\nu_2)=\sum_\mu\frac{1}{2^{\Gamma(\mu)}}|K_\mu(\nu_1)-K_\mu(\nu_2)|,$$

Γ being any injective function $V\to\mathbb{N}$. Through this metric, V is seen as a subset of the space of functions from V to $[0,1]$ with the pointwise convergence topology. Thus by Tychonoff's Theorem the completion $\tilde{V}$ of V with respect to D is a compact space, and by construction K_μ extends continuously on this completion. Actually the completion is exactly the set of sequences $(\nu_n)_{n\geq 1}$ such that for each μ, $K_\mu(\nu_n)$

converges, with two sequences $(\nu_n^1)_{n\geq 1}$, $(\nu_n^2)_{n\geq 1}$ being identified whenever for each μ, $K_\mu(\nu_n^1)$ and $K_\mu(\nu_n^2)$ have the same limit.
Denote by $\partial_M\mathcal{G}$ the set $\tilde{V}\setminus V$. The latter is called the Martin entrance boundary of the graded graph $\mathcal{G}$ and is a compact subset of $\tilde{V}$. Each element $\omega = \lim_{n\to\infty} \nu_n$ in $\partial_M\mathcal{G}$ defines a function on V by the formula

$$\omega(\mu) = \lim K_\mu(\nu_n).$$

The following Theorem is a special case of a Theorem from Doob ([4]).

Theorem 1. *With the notations above, the two following results hold:*

- *There exists a Borel subset $\partial_{\min}\mathcal{G} \subset \partial_M\mathcal{G}$ (called minimal boundary) such that for any measure α harmonic with respect to P^t, there exists a unique measure λ_α on $\partial_{\min}\mathcal{G}$ giving the kernel representation*

$$\alpha(\mu) = \int_{\partial_{\min}\mathcal{G}} K_\mu(x)d\lambda_\alpha(x).$$

- *For any reverse random walk $(X_n)_{n\leq 0}$ that respects P^t, the path $(X_0, X_{-1}, \dots)$ converges almost surely to a $\partial_{\min}\mathcal{G}-$valued random variable $X_{-\infty}$. Moreover the probability that $(X_n)_{n\leq 0}$ goes through μ is exactly $d(\mu)\mathbb{E}(K_\mu(X_{-\infty}))$.*

There exists a more general construction of the Martin boundary from Kunita and Watanabe in [14], which encompasses the case of discrete random walks as well as more general Markov processes (including the Brownian motion on a domain). However our situation is much simpler and the previous Theorem is enough.
To summarize, the Martin entrance boundary gives a topological framework to represent harmonic measures, whereas the minimal entrance boundary gives the corresponding measure theoretic framework. The situation is simpler when the two boundaries coincide. In the case of the graph $\mathcal{Z}$ that we are studying, the minimal entrance boundary was already described by Gnedin and Olshanski in [8]. The purpose of the present paper is to extend this desciption to the Martin entrance boundary by proving that the two boundary coincide.

3. The graph $\mathcal{Z}$

This section is devoted to an introduction to the graph $\mathcal{Z}$ and its relation with sequences of permutations.

3.1. **Compositions.** Let us first recall the definition of a composition:

Definition 1. *Let $n \in \mathbb{N}$. A composition λ of n, written $\lambda \vdash n$, is a sequence of positive integers $(\lambda_1, \dots, \lambda_r)$ such that $\sum \lambda_j = n$.*

Let D_λ be the subset of $[1;n]$ defined by $D_\lambda = \{\lambda_1, \lambda_1+\lambda_2, \dots, \sum_1^{r-1}\lambda_i\}$. Since there is a bijection between subsets of $[1;n-1]$ and compositions of n, D_λ is often simply denoted by λ.
To a composition is also associated a unique ribbon Young diagram with n cells:

each row j has λ_j cells, and the first cell of the row $j+1$ is just below the last cell of the row j. For example the composition $(3,2,4,1)$ of 10 is represented in Figure 1.

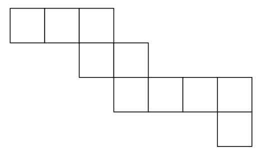

FIGURE 1. Skew Young tableau associated to the composition $\lambda = (3,2,4,1)$.

The size n is included in the definition of composition itself, since n is equal to the sum of all λ_j. If we want to emphasize the size of a compostion λ, we denote it as $|\lambda|$. When nothing is specified, λ is always assumed to have the size n, and n always denotes the size of the composition λ.
A standard filling of a composition λ of size n is a standard filling of the associated ribbon Young diagram: it is the assignement of an integer from 1 to n to each cell of the composition, such that every cells have different entries, and the entries are increasing to the right along the rows and decreasing to the bottom along the columns. An example for the composition of Figure 1 is shown in Figure 2.

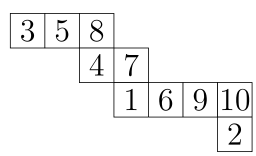

FIGURE 2. Standard filling of the composition $(3,2,4,1)$.

In particular, reading the tableau from left to right and from top to bottom gives for each standard filling a permutation σ; moreover the descent set $des(\sigma)$ of σ, namely the set of indices i such that $\sigma(i+1) < \sigma(i)$, is exactly the set D_λ. There is a bijection between the standard fillings of λ and the permutations of $|\lambda|$ with descent set D_λ. For example the filling in Figure 2 yields the permutation $(3,5,8,4,7,1,6,9,10,2)$.

3.2. **The graph $\mathcal{Z}$.** The graded graph $\mathcal{Z}$, which was introduced by Viennot in [22], is defined as follows:

(1) The set $\mathcal{Z}_n$ of vertices of degree n of $\mathcal{Z}$ is the set of compositions of n. The vertex of degree 0 is denoted $\emptyset$.
(2) Let $\lambda = (\lambda_1,\dots,\lambda_r)$ and $\mu = (\mu_1,\dots,\mu_s)$ be two compositions. There is an edge between μ and λ if and only if $|\lambda| = |\mu| + 1$ and
   - either $r = s$ and for each i except one $\mu_i = \lambda_i$ (thus exactly one μ_{i_0} is increased by one)

- either $r = s + 1$, and there exists j such that: for $k < j$, $\lambda_k = \mu_k$, $\lambda_j + \lambda_{j+1} - 1 = \mu_j$, and for $k > j$, $\lambda_{k+1} = \mu_k$ (namely one μ_i is split, and one cell is added at the end of the first piece).

The first four levels of $\mathcal{Z}$ are displayed in Figure 3.

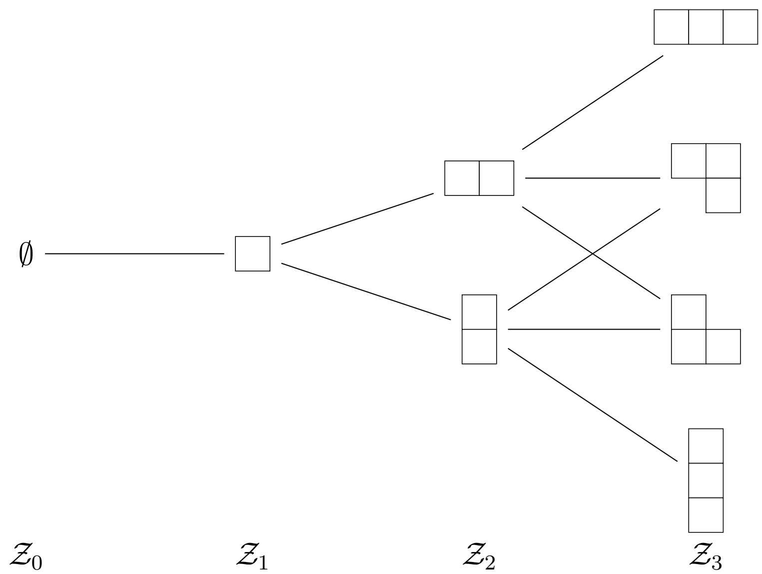

FIGURE 3. Vertices of $\mathcal{Z}$ of degree 0 to 3.

For a composition λ, let Ω_λ be the set of paths between $\emptyset$ and λ. It has been shown in [22] that $\Omega_\lambda \simeq \{\sigma \in S_{|\lambda|}, des(\sigma) = D_\lambda\}$. One way to see this is to remark that Ω_λ is the set of all standard fillings of the ribbon diagram associated to λ. Thus these sets have same cardinality and

$$d(\lambda) = |\Omega_\lambda| = \#\{\sigma \in S_{|\lambda|}, des(\sigma) = D_\lambda\}.$$

Let $\mathbb{P}_\lambda$ denote the uniform distribution on Ω_λ; from Section 2, this is equivalent to considering the random walk starting at λ with transition matrix P^t. This random walk gives n random variables σ_k^λ, $1 \leq k \leq n$, each of them being the random path restricted to the vertices of degree smaller than k.

Since there is a bijection between paths on $\mathcal{Z}$ from $\emptyset$ to μ and permutations of $|\mu|$ with descent set D_μ, each σ_k^λ is a random permutation in S_k, and the law of $\sigma_\lambda = \sigma_n^\lambda$ is the uniform distribution on the set of permutations with descent set D_λ. Moreover a counting argument yields that for $\sigma \in \mathfrak{S}_k$ with $des(\sigma) = D_\mu$, under the probability $\mathbb{P}_\lambda$,

$$\mathbb{P}_\lambda(\sigma_\lambda^k = \sigma) = \frac{d(\mu, \lambda)}{d(\lambda)}. \tag{2}$$

By abuse of notation a finite path starting at $\emptyset$ on $\mathcal{Z}$ and the corresponding permutation are both usually denoted by σ. In particular if $\sigma \in \Omega_\lambda$, σ_k denotes the path

after k steps, whereas $\sigma(i)$ will denote the image of i by the permutation associated to σ (the same for $\sigma(A)$ with A a subset of $\{1,\dots,n\}$).

3.3. **Arrangement on $\mathbb{N}$.** In this paragraph a permutation $\sigma \in \mathfrak{S}_k$ is written as a word in the alphabet $\{1,\dots,k\}$, where $i_j = \sigma(j)$. For $k \geq 2$ and $\sigma = (i_1 \dots i_k)$, $\sigma_\downarrow \in \mathfrak{S}_{k-1}$ is defined as the permutation $(i_1 \dots \cancel{k} \dots i_k)$. If $\sigma \in \mathfrak{S}_n$, $\sigma_{\downarrow k}$ denotes the $(n-k)-$iteration of the $\downarrow$-operation : namely all the indices between $k+1$ and n have been erased.
An arrangement of $\mathbb{N}$ is a sequence $(\sigma_1,\dots,\sigma_k,\dots)$ such that for all $k \geq 1$, $\sigma_k \in \mathfrak{S}_k$, and such that the following compatibility condition holds :

$$(\sigma_k)_\downarrow = \sigma_{k-1}.$$

For example the following sequence is the first part of an arrangement :

$$((1),(21),(231),(2341),(52341),\dots).$$

The set of all arrangements is denoted $\mathcal{A}$. For $k \geq 1$, let $\pi_k : \mathcal{A} \to \mathfrak{S}_k$ be the application which consists in the projection of the sequence $(\sigma_1,\sigma_2,\dots)$ on the $k-$th element σ_k. $\mathcal{A}$ is considered with the initial topology with respect to the applications π_k, and with the corresponding borelian $\sigma-$algebra. Thus by the Kolmogorov's extension Theorem, any random variable Π on $\mathcal{A}$ is uniquely determined by the law of its finite-dimensional projections $(\pi_1(\Pi),\dots,\pi_k(\Pi))$.
The result of the previous subsection yields that there is a bijection between infinite paths on $\mathcal{Z}$ and arrangements of $\mathbb{N}$, and from Section 2 this bijection extends to a bijection between harmonic measures α with respect to P^t and random arrangements Π such that

$$\mathbb{P}(\pi_1(\Pi) = \sigma_1,\dots,\pi_k(\Pi) = \sigma_k) = p(des(\sigma_k)),$$

with p a positive function on $\mathcal{Z}$ given by $p = \frac{\alpha}{d}$. This correspondance is convenient since it allows to describe the solutions of the problem (1) in terms of random arrangements.

4. Paintbox construction and Minimal boundary

Thanks to the latter correspondance, Gnedin and Olshanski described the minimal entrance boundary of $\mathcal{Z}$ in terms of random arrangements. This description is the purpose of the following paragraph.

4.1. **Paintbox construction.** The description is based on a topological space consisting in pairs of disjoint open sets of $[0,1]$:

Definition 2. *The topological space $\mathcal{U}^{(2)}$ is the space*

$$(\{(U_\uparrow,U_\downarrow)|U_\uparrow \textit{ and } U_\downarrow \textit{ disjoint open sets of }]0,1[\},d),$$

with the distance d between $(U_\uparrow, U_\downarrow)$ and $(V_\uparrow, V_\downarrow)$ given by

$$d((U_\uparrow, U_\downarrow), (V_\uparrow, V_\downarrow)) = \sup(d_{Haus}(U_\uparrow^c, V_\uparrow^c), d_{Haus}(U_\downarrow^c, V_\downarrow^c)).$$

Let $M_1(\mathcal{U}^{(2)})$ denote the set of probability measures with respect to the $\sigma-algebra$ coming from the above topology.
From the definition of the metric, $(U_\uparrow(j), U_\downarrow(j)))_{j\geq 1}$ converges to $(V_\uparrow, V_\downarrow))$ if and only if for each $\epsilon > 0$:

- for j large enough, the number of connected components of size larger than ϵ in $U_\uparrow$ and $V_\uparrow$ are the same,
- the boundaries of the connected components of size larger than ϵ in $U_\uparrow$ converge to the ones of $V_\uparrow$,
- the same holds by switching $\uparrow$ and $\downarrow$.

In particular $(U_\uparrow(j), U_\downarrow(j))$ converges to $(\emptyset, \emptyset)$ if and only if the size of the largest components in $U_\uparrow(j)$ and $U_\downarrow(j)$ tends to 0. The following important result holds for $\mathcal{U}^{(2)}$:

Proposition 1. *$\mathcal{U}^{(2)}$ is compact space.*

The minimal entrance boundary of $\mathcal{Z}$ is described by random arrangements constructed from elements of $\mathcal{U}^{(2)}$.

Definition 3. *Let $U = (U_\uparrow, U_\downarrow)$ be fixed, $(X_1, \dots, X_k, \dots)$ a sequence of $[0,1]$. For each $k \geq 1$, $\sigma_U(X_1, \dots, X_k) \in \mathfrak{S}_k$ is defined by the following rule:*
$(\sigma_U(X_1, \dots, X_k))^{-1}(i)$ is less than $(\sigma_U(X_1, \dots, X_k))^{-1}(j)$ if and only if one of the three following situations arises :

- *X_i and X_j are not in the same connected component of $U_\uparrow$ or $U_\downarrow$ and $X_i < X_j$*
- *X_i and X_j are in the same connected component of $U_\uparrow$ and $i < j$*
- *X_i and X_j are in the same connected component of $U_\downarrow$ and $j < i$.*

The random variable $\sigma_U(X_1, \dots, X_k)$ defined for an infinite family $(X_1, \dots, X_k, \dots)$ of independent uniform variables on $[0,1]$ is denoted $\sigma_U(k)$. The sequence $(\sigma_U(1), \sigma_U(2), \dots)$ is denoted σ_U.

The construction of $\sigma_U(X_1, \dots, X_k)$ from $(X_1, \dots, X_k)$ and $U \in \mathcal{U}^{(2)}$ is well-defined and unique. If $U = (\emptyset, \emptyset)$, $\sigma_{(\emptyset,\emptyset)}(X_1, \dots, X_k)$ is just the permutation associated to the reordering $(X_{i_1} < X_{i_2} < \dots X_{i_k})$. This permutation is denoted by $Std^{-1}(X_1, \dots, X_k)$. For each k, the random variable $\sigma_{(\emptyset,\emptyset)}(k)$ has a uniform distribution on $\mathfrak{S}_k$.
The next Theorem is due to Gnedin and Olshanski in [8] (based on an important work of Jacka and Warren in [10]) and identify $\mathcal{U}^{(2)}$ with the minimal entrance boundary of the graded graph $\mathcal{Z}$:

Theorem 2. *Each random variable σ_U defines a random arrangement $\mathcal{A}$ that comes from an harmonic probability measure on $(\mathcal{Z}, P^t)$, and there is an isomorphism :*

$$\Phi : M_1(\mathcal{U}^{(2)}) \longrightarrow M_1(\partial_{\min}\mathcal{Z})$$

which restricts to a bijective map $p : \mathcal{U}^{(2)} \longrightarrow \partial_{\min}\mathcal{Z}$ mapping $\delta_{(U_\uparrow,U_\downarrow)}$ to $\sigma_{(U_\uparrow,U_\downarrow)}$.

In particular for each $k \geq 1$ and $\sigma \in \mathfrak{S}_k$, $\mathbb{P}(\sigma_U(k) = \sigma)$ only depends on the descent set μ of σ and is thus denoted by $p_U(\mu)$.

4.2. **Martin entrance boundary of $\mathcal{Z}$.** The question is to know if $\partial_{\min}\mathcal{Z} = \partial_M\mathcal{Z}$. The problem is summed up in Conjecture 45 of [8]. To each composition λ of n is associated an element $U_\lambda = (U_\uparrow(\lambda), U_\downarrow(\lambda))$ of $\mathcal{U}^{(2)}$ as follows : for each $s \leq n-1$ set $I_s = [\frac{s-1}{n-1}, \frac{s}{n-1}]$, and define

$$U_\uparrow(\lambda) = int(\bigcup_{i \notin des(\lambda)} I_s), U_\downarrow(\lambda) = int(\bigcup_{i \in des(\lambda)} I_s),$$

with *int* denoting the interior of a set. Then the conjecture states the following :

Conjecture 1. a) *A sequence $(\lambda_n)_{n\geq 1}$ is in $\partial_M\mathcal{Z}$ if and only if U_{λ_n} converges in $\mathcal{U}^{(2)}$.*
b) *$U_{\lambda_n} \to_{\mathcal{U}^{(2)}} (U_\uparrow, U_\downarrow)$ is equivalent to $K_\mu(\lambda_n) \to p_{(U_\uparrow,U_\downarrow)}(\mu)$ for all $\mu \in \mathcal{Z}$.*
c) *The Martin boundary of the graph $\mathcal{Z}$ actually coincides with its minimal boundary : $\partial_M\mathcal{Z} = \mathcal{U}^{(2)}$*

Actually, the only difficult part is to prove the first implication of b):

$$(3) \qquad (U_{\lambda_n} \to_{\mathcal{U}^{(2)}} (U_\uparrow, U_\downarrow)) \Longrightarrow (\forall \mu \in \mathcal{Z}, K_\mu(\lambda_n) \to p_{(U_\uparrow,U_\downarrow)}(\mu)).$$

Indeed suppose that the latter is true :

Proof. *a)* Let $\omega = (\lambda_n)_{n\geq 1}$ be in $\partial_M\mathcal{Z}$. Since $\mathcal{U}^{(2)}$ is compact, proving the convergence of U_{λ_n} in $\mathcal{U}^{(2)}$ is the same as proving that every convergent subsequences of U_{λ_n} have the same limit. Let $(\lambda_{\phi(n)})_{n\geq 1}$ and $(\lambda_{\phi'(n)})_{n\geq 1}$ be such that

$$U_{\lambda_{\phi(n)}} \to (U^1_\uparrow, U^2_\downarrow), U_{\lambda_{\phi'(n)}} \to (U^2_\uparrow, U^2_\downarrow)$$

Then by (3), for all $\mu \in \mathcal{Z}$, $\omega(\mu) = p_{U^1_\uparrow,U^1_\downarrow}(\mu)$ and $\omega(\mu) = p_{U^2_\uparrow,U^2_\downarrow}(\mu)$. Since $p : \mathcal{U}^{(2)} \to \partial_{\min}\mathcal{Z}$ is injective, necessarily $(U^1_\uparrow, U^1_\downarrow) = (U^2_\uparrow, U^2_\downarrow)$. This shows that U_{λ_n} converges.
Conversely if U_{λ_n} converges in $\mathcal{U}^{(2)}$, the assumption (3) implies directly that $(\lambda_n) \in \partial_M\mathcal{Z}$.
b) The direct implication is exactly (3); for the converse implication, the convergence of $K_\mu(\lambda_n)$ for all $\mu \in \mathcal{Z}$ implies that $(\lambda_n)_{n\geq 1} \in \partial_M\mathcal{Z}$. Thus from a), U_{λ_n} converges in $\mathcal{U}^{(2)}$. By injectivity of p, U_{λ_n} converges to $(U_\uparrow, U_\downarrow)$.
c) This is the summary of 1) and 2).

□

The following sections are devoted to the proof of the implication (3), which implies Conjecture 1 :

Theorem 3. *Let λ_n be a sequence of compositions such that $\lambda_n \vdash n$. If U_{λ_n} converges to $(U_\uparrow, U_\downarrow)$, then for all $\mu \in \mathcal{Z}$,*

$$K_\mu(\lambda_n) \to p_{(U_\uparrow, U_\downarrow)}(\mu).$$

The proof of this theorem is based on the construction, for each composition λ, of a family of random variables, that mimics the kernel $K_\mu(\lambda)$. The construction of this family is done in the next section. Theorem 3 is then deduced from the fact that this family of random variables becomes close to a family of independent uniform random variables when λ becomes large, allowing to recover the Paintbox construction and thus the link with the minimal boundary. The convergence to the family of uniform random variables is not clear and explained in Section 7 and 8. The proof uses the results of a previous paper, [19], that deals with combinatoric of large compositions, and these results are summarized in section 6.

5. The familiy $(\xi_i^\lambda)_{i\geq 1}$

Some definitions on compositions are needed before defining the family $(\xi_i^\lambda)_{i\geq 1}$.

5.1. **Combinatoric of compositions.** Let λ be a composition of n with its associated descent set $D_\lambda = (\lambda_1, \lambda_1+\lambda_2, \dots, \lambda_1+\cdots+\lambda_{r-1})$. An integer $i \in [1;n]$ is a peak of λ if $i \in D_\lambda \cup \{n\}$ and $i-1 \notin D_\lambda$, and $i \in [1;n]$ is a valley if $i \notin D_\lambda$ and $i-1 \in D_\lambda \cup \{0\}$. This definition makes sense if we consider any standard filling σ of λ : $\sigma(i)$ is a local maximum (resp. minimum) of $\sigma = \sigma(1)\dots\sigma(n)$ if and only if i is a peak (resp. valley) of λ. Let V denote the set of valleys, P the set of peaks, and $\mathcal{E} = V \cup P$ the set of extreme cells.
A run s of λ is an interval $[a;b]$ of $[1;n]$ such that a, b are consecutive integers of $\mathcal{E}$. A run $[a;b]$ is called descending if $a \in P$ and ascending if $a \in V$. The runs are ordered by the lower endpoint of the corresponding interval, and this yields a total ordered set $S = \{s_i\}_{1\leq i\leq t}$. Each element s_i of S corresponds to an interval $[a_i; a_{i+1}]$, with $a_1 = 1$ and $a_{t+1} = n$. In particular two consecutive runs s_i and s_{i+1} overlap on a_{i+1}. The length of a run s_i is defined as the value $l_i = a_{i+1} - a_i$. For example if $\lambda = (3,2,4,1)$, $V = \{1,4,6,10\}$, $P = \{3,5,9\}$ and $S = \{[1;3],[3;4],[4;5],[5,6],[6;9],[9,10]\}$.
For any cell i of λ, the slope of i, $\mathfrak{s}(i) = [x(i); y(i)]$, is defined as the maximum subinterval of $[1;n]$ that contains i and no other peak or valley. In the previous example, $\mathfrak{s}(7) = [7;8]$ and $\mathfrak{s}(6) = [6;8]$.

5.2. **Definition of $(\xi_i^\lambda)_{i\geq 1}$.** Let $(X^i(p,q))_{\substack{i\geq 1\\ \{p,q\}\subset\mathbb{Q}}}$ be a family of independent variables such that $X^i(p,q) \sim Unif([p,q])$.

Definition 4. *Let λ be a composition of n, $\sigma \in \Omega_\lambda$. The averaged coordinate of i with respect to σ is the random variable defined by $\xi_i(\sigma) = 0$ if $i > n$, and*

$$\xi_i(\sigma) = X^i\Big(\frac{x(\sigma^{-1}(i)) - 1}{n}, \frac{y(\sigma^{-1}(i))}{n}\Big),$$

for $1 \leq i \leq n$.
For σ_λ *chosen uniformly among* Ω_λ, $\xi_i(\sigma_\lambda)$ *is denoted* ξ_i^λ. $\xi^\lambda(k)$ *denotes the vector* $(\xi_1^\lambda, \ldots, \xi_k^\lambda)$ *and* $\xi^\lambda(n)$ *is simply written* ξ^λ.

Basically constructing ξ_i^λ means that we sample a uniformly random standard filling σ_λ of λ, we look at the cell containing i with respect to this filling, and then sample a random variable uniformly distributed on the rescaled slope of this cell. The advantage is that the knowledge of $\xi^\lambda(k)$ is enough to reconstruct σ_k^λ. This reconstruction needs a slightly modified version of U_λ:

Definition 5. *The run paintbox* $\tilde{U}_\lambda$ *associated to* λ *is an element of* $\mathcal{U}^{(2)}$ *consisting in the following open subsets:*

- $\tilde{U}_\uparrow(\lambda) = \bigcup_{a_i \in V}]\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}[$
- $\tilde{U}_\downarrow(\lambda) = \bigcup_{a_i \in P}]\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}[$

with $a_{i+1} = n+1$ *if* $a_i = n$.

The run paintbox $\tilde{U}_\lambda$ becomes close to U_λ when n goes to infinity:

Lemme 1. *Let* λ *be a composition of* n. *With respect to the distance on* $\mathcal{U}^{(2)}$,

$$d(U_\lambda, \tilde{U}_\lambda) \leq \frac{1}{n}.$$

Proof. The definition of U_λ yields the following open sets:

$$U_\uparrow(\lambda) = \bigcup_{a_i \in V, a_i \neq n}]\frac{a_i-1}{n-1}, \frac{a_{i+1}-1}{n-1}[,$$

and

$$U_\downarrow(\lambda) = \bigcup_{a_i \in P, a_i \neq n}]\frac{a_i-1}{n-1}, \frac{a_{i+1}-1}{n-1}[.$$

Let us show that $U_\uparrow(\lambda)^c$ is included in the $\frac{1}{n}-$inflation of $\tilde{U}_\uparrow(\lambda)^c$ and conversely (the proof for $U_\downarrow(\lambda)^c$ and $\tilde{U}_\downarrow(\lambda)^c$ is the same). The $\frac{1}{n}-$inflation of $U_\uparrow(\lambda)^c$ is

$$U_\uparrow(\lambda)^{c,1/n} = \left(\bigcup_{a_i \in P, a_i \neq n} [(\frac{a_i-1}{n-1} - \frac{1}{n}) \vee 0, (\frac{a_{i+1}-1}{n-1} + \frac{1}{n}) \wedge 1] \right) \cup [0, \frac{1}{n}] \cup [1 - \frac{1}{n}, 1].$$

On the other hand

$$\tilde{U}_\uparrow(\lambda)^c = \left(\bigcup_{a_i \in P} [\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}] \right) \cup \{0\} \cup \{1\}.$$

Suppose that $a_i \neq n$. Then for all $1 \leq k \leq n-1$, $\frac{k}{n-1} - \frac{1}{n} \leq \frac{k}{n} \leq \frac{k}{n-1} + \frac{1}{n}$, thus

$$[\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}] \subset [(\frac{a_i-1}{n-1} - \frac{1}{n}) \vee 0, (\frac{a_{i+1}-1}{n-1} + \frac{1}{n}) \wedge 1] \subset U_\uparrow(\lambda)^{c,1/n}.$$

If $a_i = n$, $[\frac{a_i-1}{n}, \frac{a_{i+1}-1}{n}] = [1-1/n, 1] \subset U_\uparrow(\lambda)^{c,1/n}$. Finally $\tilde{U}_\uparrow(\lambda)^c \subset U_\uparrow(\lambda)^{c,1/n}$.
For the converse inclusion the $\frac{1}{n}-$inflation of $\tilde{U}_\uparrow(\lambda)^c$ is

$$\tilde{U}_\uparrow(\lambda)^{c,1/n} = \left(\bigcup_{a_i\in P}[(\frac{a_i-1}{n}-\frac{1}{n})\vee 0, (\frac{a_{i+1}-1}{n}+\frac{1}{n})\wedge 1]\right)\cup[0,\frac{1}{n}]\cup[1-\frac{1}{n},1],$$

and

$$U_\uparrow(\lambda)^c = \left(\bigcup_{a_i\in P, a_i\neq n}[\frac{a_i-1}{n-1},\frac{a_{i+1}-1}{n-1}]\right)\cup\{0\}\cup\{1\}.$$

Since for $1\leq k\leq n-1$, $\frac{k}{n}-\frac{1}{n}\leq\frac{k}{n-1}\leq\frac{k}{n}+\frac{1}{n}$, for each $a_i\neq n$,

$$[\frac{a_i-1}{n-1},\frac{a_{i+1}-1}{n-1}]\subset[(\frac{a_i-1}{n}-\frac{1}{n})\vee 0,(\frac{a_{i+1}-1}{n}+\frac{1}{n})\wedge 1],$$

and therefore $U_\uparrow(\lambda)^c\subset\tilde{U}_\uparrow(\lambda)^{c,1/n}$.
Doing the same for $U_\downarrow(\lambda)$ and $\tilde{U}_\downarrow(\lambda)$ concludes the proof. □

The previous Lemma implies that for any sequence $(\lambda_n)_{n\geq 1}$ with $|\lambda_n|\to\infty$, the convergence of U_{λ_n} is equivalent to the convergence of $\tilde{U}_{\lambda_n}$, and both have the same limit. The advantage is that the knowledge of ξ_k^λ and $\tilde{U}_{\lambda_n}$ is enough to recover σ_k^λ.

Proposition 2. *For each $1\leq k\leq n$, $\sigma\in\Omega_\lambda$,*

$$\sigma_{\tilde{U}_\lambda}((\xi_i(\sigma))_{1\leq i\leq k}) = \sigma_{\downarrow k}.$$

In particular the random variables σ_k^λ and $\sigma_{\tilde{U}_\lambda}(\xi^\lambda(k))$ have the same law.

Proof. It is enough to prove it for $k=n$. Denote $\xi_i(\sigma)=\xi_i$ and $\xi=(\xi_i(\sigma))_{1\leq i\leq n}$. It is equivalent to prove that for $1\leq i,j\leq n$,

$$\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j)\Leftrightarrow\sigma_\lambda^{-1}(i)<\sigma_\lambda^{-1}(j).$$

Let $1\leq i<j\leq n$. Then $\sigma_\lambda^{-1}(i)<\sigma_\lambda^{-1}(j)$ implies that i is left to j in the associated filling of λ. This is possible in one of the two following situations :

(1) $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ are disjoint and $\mathfrak{s}(i)$ is left to $\mathfrak{s}(j)$. In this case ξ_i and ξ_j are not in the same interval component of $\tilde{U}_\lambda$ and ξ_i is in an interval component left to the one of ξ_j. By the run Paintbox construction,

$$\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i)<\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j).$$

(2) $\mathfrak{s}(i)$ and $\mathfrak{s}(j)$ overlap. This implies that i and j are in a same run $s=[a_i;a_{i+1}]$ of λ. Let $I_s=]\frac{a_i-1}{n},\frac{a_{i+1}-1}{n}[$. Since $i<j$ and $\sigma_\lambda^{-1}(i)<\sigma_\lambda^{-1}(j)$, the run s has to be an ascending one and thus $a_i\in V$ and $a_{i+1}\in P$. In particular $\sigma_\lambda^{-1}(i)$ cannot be a peak, and $\sigma_\lambda^{-1}(j)$ cannot be a valley. Thus ξ_i is either in an interval component left to I_s, either in I_s. For similar reasons, ξ_j is either in an interval component right to I_s, either in $_s$. This implies that if ξ_i or ξ_j

is not in I_s, $\left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j)$. But if ξ_i and ξ_j are both in I_s, since the latter is in $\tilde{U}_\uparrow(\lambda)$, the same inequality holds.

Finally, in any case,

$$\sigma_\lambda^{-1}(i) < \sigma_\lambda^{-1}(j) \Longrightarrow \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) < \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j).$$

The proof is exactly the same to prove that

$$\sigma_\lambda^{-1}(i) > \sigma_\lambda^{-1}(j) \Longrightarrow \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(i) > \left(\sigma_{\tilde{U}_\lambda}(\xi)\right)^{-1}(j),$$

yielding the first part of the Proposition. This first part implies clearly the second one. ☐

It is also possible to recover exactly the position of $\{1, \ldots, k\}$ in the filling σ of λ from $(\xi_i(\sigma))_{1\leq i\leq k}$:

Lemme 2. *Let σ, σ' be two permutations of Ω_λ. If $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k))$ is not equal to $(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k))$, then $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports.*

Proof. The proof is done by recurrence on $k \geq 1$. Let $k = 1$. 1 has to be located in a valley of λ. If $\sigma^{-1}(1) \neq \sigma'^{-1}(1)$, 1 is located in a different valley in σ and σ'. Thus the slopes of $\sigma^{-1}(1)$ and $\sigma'^{-1}(1)$ are disjoint, and $\xi_1(\sigma)$ and $\xi_1(\sigma)$ have disjoint supports.
Let $k > 1$. Suppose that $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k)) \neq (\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k))$. By recurrence hypothesis, if $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1))$ is not equal to $(\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k-1))$, $(\xi_1(\sigma), \ldots, \xi_{k-1}(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_{k-1}(\sigma'))$ have disjoint supports. This yields also that $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports.
Thus let us assume that $(\sigma^{-1}(1), \ldots, \sigma^{-1}(k-1)) = (\sigma'^{-1}(1), \ldots, \sigma'^{-1}(k-1))$. This implies that $\sigma^{-1}(k) \neq \sigma'^{-1}(k)$; since the position of $\{1, \ldots, k-1\}$ is the same in the fillings σ and σ' of λ, the cell containing k in σ is in a different run than the cell containing k in σ'. Therefore their slopes are disjoint, and $(\xi_1(\sigma), \ldots, \xi_k(\sigma))$ and $(\xi_1(\sigma'), \ldots, \xi_k(\sigma'))$ have disjoint supports. ☐

This section ends by a convergence result, whose proof is left to the appendix.

Proposition 3. *Let U_n be a sequence of $\mathcal{U}^{(2)}$ and $((X^n(i))_{i\geq 1})_{n\geq 1}$ a sequence of random infinite vectors on $[0, 1]$. Let $(X^0(1), \ldots, X^0(n), \ldots)$ be a random infinite vector on $[0, 1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_n \to U \in \mathcal{U}^{(2)}$ and for each $k \geq 1$, $X_k^n = (X^n(1), \ldots, X^n(k))$ converges in law to $X_k^0 = (X^0(1), \ldots, X^0(k))$, then for each $k \geq 1$,*

$$\sigma_{U_n}(X_k^n) \longrightarrow_{law} \sigma_U(X_k^0).$$

6. Combinatoric of large compositions

The purpose of this section is to introduce the background material to prove that the family $(\xi_u^\lambda)_{1\le u\le k}$ converges in law to a family of independent uniform random variables on $[0,1]$. Since ξ_u^λ depends uniquely on the runs in which u is located in a random filling σ_λ of λ, it is necessary to evaluate the probability for u to be located in a particular run s of λ. For a composition λ and $i\in\lambda$ a fixed cell, denote by $\lambda_{\le i}$ (resp. $\lambda_{\ge i}$, $\lambda_{<i}$, $\lambda_{>i}$) the composition λ restricted to cells left (resp. right, res. strictly left, resp. strictly right) to i. Recall that $d(\lambda)$ denotes the number of standard fillings of the ribbon Young diagram associated to λ.
Let us focus here on the location of 1 in σ_λ. Since 1 is necessary a local minimum in any filling of λ, it has to be located in a valley $v\in V$. Thus a direct counting argument shows that for a fixed valley v of λ,

$$\mathbb{P}_{\sigma_\lambda}(1\in v)=\frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!}\frac{d(\lambda_{>v})d(\lambda_{<v})}{d(\lambda)}. \tag{4}$$

The problem is therefore essentially to relate $d(\lambda_{>v})d(\lambda_{<v})$ to $d(\lambda)$.

6.1. **Probabilistic approach to descent combinatorics.** Ehrenborg, Levin and Readdy (see [7]) formalized in the context of descent sets an old relation between permutations of n and polytopes in $[0,1]^n$. Namely since the volume of the set $R_\sigma=\{x_{\sigma(1)}<\cdots<x_{\sigma(n)}\}$ is exactly $\frac{1}{n!}$, it is possible to determine probabilistic quantities on $\mathfrak{S}_n$ by integrating certain functions that are constant on each region R_σ. The appropriate functions for descent sets were found in [7], yielding some new estimates as in [6] and [1]. The model of Ehrenborg, Levin and Readdy is exposed in this paragraph, but in a modified way to focus only on the set of extreme cells $\mathcal{E}$ (as defined in the paragraph 5.1). This yields the following framework : let λ be a composition of $n\ge 2$ with set of extreme cells $\mathcal{E}=\{a_1=1,a_2,\dots,a_r=n\}$. Suppose for example that the first cell is a valley (namely $a_1\in V$) and denote by s_j the run between a_j and a_{j+1}, with l_j its length. To each λ is associated the couple of random variable (X_λ,Y_λ) on $[0,1]^2$ with density

$$d_{X_\lambda,Y_\lambda}(x_1,x_r)=\frac{1}{\mathcal{V}_\lambda}\int_{[0,1]^{r-2}}\prod\mathbf{1}_{x_{2i-1}<x_{2i}>x_{2i+1}}\prod_{1\le i\le r-1}\frac{|x_i-x_{i+1}|^{l_i-1}}{(l_i-1)!}\prod_{2\le i\le r-1}dx_i. \tag{5}$$

If the first cell is a peak (i.e $a_1\in P$), the inequalities in the expression of the density are reversed. If $\lambda=\square$, the expression for the distribution of $(X_\square,Y_\square)$ (in the distributional sense) is simply:

$$d_{X_\square,Y_\square}(u,v)=\delta_{u=v}.$$

The latter probabilistic model is related to the combinatoric of descent sets through the equality

$$d(\lambda)=|\lambda|!\mathcal{V}_\lambda, \tag{6}$$

whose proof can be found in [7].
The first advantage of this model is that it behaves simply with respect to concatenation of compositions.

Definition 6. *Let $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ be two compositions of m and n. The concatenated composision $\lambda + \mu$ is the composition of $n + m$*

$$\lambda + \mu = (\lambda_1, \dots, \lambda_r + \mu_1, \mu_2, \dots, \mu_s),$$

and the concatenated composition $\lambda - \mu$ is the composition of $n + m$

$$\lambda + \mu = (\lambda_1, \dots, \lambda_r, \mu_1, \mu_2, \dots, \mu_s).$$

This definition has a simple meaning in terms of associated ribbon Young diagrams: namely the diagram of $\lambda + \mu$ (resp. $\lambda - \mu$) is the juxtaposition of the one of λ and the one of μ such that the last cell of λ is left to (resp. above) the first cell of μ. An application of the section 2 of [7] (see also Lemma 2 in [1]) implies the following expression of the concatenation in the probabilistic framework :

Proposition 4. *Let λ, μ be two compositions, $\epsilon \in \{-, +\}$. Then*

$$\mathcal{V}_{\lambda\epsilon\mu} = \mathcal{V}_\lambda \mathcal{V}_\mu \mathbb{E}(Y_\lambda \leq_\epsilon X_\mu)$$

and

$$d_{X_{\lambda\epsilon\mu}, Y_{\lambda\epsilon\mu}}(x, y) = \frac{1}{\mathbb{E}(Y_\lambda \leq_\epsilon X_\mu)} \int_{[0,1]^2} d_{X_\lambda, Y_\lambda}(x, u) \mathbf{1}_{u \leq_\epsilon v} d_{X_\mu, Y_\mu}(v, y) du dv,$$

where $\leq_- = \geq$ and $\leq_+ = \leq$, and the couples (X_λ, Y_λ) and (X_μ, Y_μ) are considered as independent.

The previous Proposition yields a particular case that helps to compute the law of ξ_1^λ. Denote by F_X the distributive cumulative function of a random variable X.

Corollary 1. *Let λ be a composition of n and v a valley of λ. Then*

$$\mathbb{P}_\lambda(1 \in v) = \frac{1}{n} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t)) dt},$$

with the convention $X_{\lambda_{>n}} = \delta_1$ and $Y_{\lambda_{<1}} = \delta_1$.

Proof. Since v is a valley, λ can be written $\lambda_{<v} - \square + \lambda_{>v}$. Thus the previous Proposition yields

$$\mathcal{V}_{\lambda_{<v} - \square + \lambda_{>v}} = \mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{\geq v}} \mathbb{E}(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}).$$

Conditioning the expectation on the value of $X_{\lambda_{\geq v}}$ gives by independence,

$$\mathbb{E}(Y_{\lambda_{<v}} \geq X_{\lambda_{\geq v}}) = \int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t)) d_{X_{\lambda_{\geq v}}}(t) dt.$$

On the other hand from the previous Proposition, since $X_{\lambda_{\geq v}} = X_{\square+\lambda_{>v}}$,

$$\begin{aligned} d_{X_{\lambda_{\geq v}}}(t) =& \frac{1}{\mathbb{E}(X_{\lambda_{>v}} \geq Y_\square)} \int_{[0,1]^3} \delta(t,u) \mathbf{1}_{u \leq v} d_{X_{\lambda_{>v}}, Y_{\lambda_{>v}}}(v,y) du dv dy \\ =& \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_\square}{\mathcal{V}_{\lambda_{\geq v}}} (1 - F_{X_{\lambda_{>v}}}(t)), \end{aligned}$$

and finally

$$\mathcal{V}_{\lambda_{<v} - \square + \lambda_{>v}} = \mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{>v}} \int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t)) dt.$$

Using the latter result in the equalities (4) and (6) yields

$$\begin{aligned} \mathbb{P}_{\sigma_\lambda}(1 \in v) =& \frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!} \frac{d(\lambda_{>v}) d(\lambda_{<v})}{d(\lambda)} \\ =& \frac{(|\lambda|-1)!}{|\lambda_{<v}|!|\lambda_{>v}|!} \frac{|\lambda_{<v}|!|\lambda_{>v}|!}{|\lambda|!} \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_\lambda} \\ =& \frac{1}{|\lambda|} \frac{\mathcal{V}_{\lambda_{>v}} \mathcal{V}_{\lambda_{<v}}}{\mathcal{V}_{\lambda_{<v}} \mathcal{V}_{\lambda_{>v}} \int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t)) dt} \\ =& \frac{1}{|\lambda|} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t)) dt}. \end{aligned}$$

□

6.2. **Estimates on** (X_λ, Y_λ)**.** The latter corollary shows that the knowledge of F_{X_μ} and F_{Y_μ} for a subcomposition μ of λ yields estimates on the location of 1 in σ_λ. The results on the behavior of F_{X_μ}, F_{Y_μ} obtained in [19] are summarized in this paragraph, and the reader should refer to this article for the corresponding proofs. The first result is a bound of F_{X_λ} depending on the length of the first run of λ and corresponds to Corollary 2 in[19]:

Proposition 5. *Let λ be a composition with at least two runs, and with first run of length R. If the first run is increasing, the following inequality holds :*

$$1 - (1-t)^R \leq F_{X_\lambda}(t) \leq 1 - (1-t)^{R+1}.$$

If the first run is decreasing, the inequality is

$$t^{R+1} \leq F_{X_\lambda}(t) \leq t^R.$$

A similar result holds for Y_λ. The latter inequalities are very accurate when R is large, but when the runs remain bounded, the result is not so useful. It is still possible to show that the distribution of X_λ only depends on the first cells of the composition. This corresponds to Proposition 11 in [19].

Proposition 6. *Let $\epsilon > 0$. There exists $n_0 \geq 1$ such that for any $n \geq n_0$ and any composition λ of size larger than n with first run smaller than n,*

$$\|F_{X_{\lambda_{\leq n}}} - F_{X_\lambda}\|_\infty \leq \epsilon.$$

In the latter result, n_0 depends only on ϵ, and not on the shape of λ.

7. Asymptotic law of ξ_1^λ

This section is devoted to the asymptotic law of ξ_1^λ.

7.1. **Preliminary results.** Propositions 5 and 6 imply that $\mathbb{P}_\lambda(1 \in v)$ only depends on the shape of λ around v.

Lemme 3. *Let $\epsilon > 0$. There exists $n_\epsilon \in \mathbb{N}$ such that for $n_0 \geq n_\epsilon$ and two compositions $\lambda \vdash n$ and $\mu \vdash m$ with the first run of μ smaller than n_0,*

$$1 - \epsilon \leq \frac{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t))dt}{\int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_{\mu_{\leq n_0}}}(t))dt} \leq 1 + \epsilon.$$

Proof. Let λ, μ be two compositions, with L the size of the last run of λ and R the size of the first run of μ. Set $\epsilon_1 = +$ if the last run of λ is increasing, $\epsilon_1 = -$ else, and the same with ϵ_2 and the first run of μ. Let $\Delta = \int_0^1 (1 - F_{Y_\lambda}(t))(1 - F_{X_\mu}(t))dt$. From Proposition 5, integrating the inequalities yields the following bounds on Δ:

- If $\epsilon_1 = +, \epsilon_2 = +$,
$$\frac{1}{R+1}(1 - \frac{1}{(R+2)\dots(R+L)}) \leq \Delta \leq \frac{1}{R}(1 - \frac{1}{(R+1)\dots(R+L)}),$$
- if $\epsilon_1 = -, \epsilon_2 = +$,
$$\frac{1}{L+R+1} \leq \Delta \leq \frac{1}{L+R-1},$$
- if $\epsilon_1 = +, \epsilon_2 = -$,
$$1 - \frac{1}{R} - \frac{1}{L} + \frac{1}{R+L-1} \leq \Delta \leq 1 - \frac{1}{R+1} - \frac{1}{L+1} + \frac{1}{L+R+1},$$
- if $\epsilon_1 = -, \epsilon_2 = -$,
$$\frac{1}{L+1}(1 - \frac{1}{(L+2)\dots(L+R)}) \leq \Delta \leq \frac{1}{L}(1 - \frac{1}{(L+1)\dots(L+R)}).$$

The latter bounds are independent of the shape of λ, μ apart from the lengths of the last run of λ and the first run of μ. Denote by $A_{L,R}^{\epsilon_1,\epsilon_2}$ each upper bound in the previous list, and $B_{L,R}^{\epsilon_1,\epsilon_2}$ each lower bound. Then as $\min(L, R) \to \infty$,

$$\frac{B_{L,R}^{\epsilon_1,\epsilon_2}}{A_{L,R}^{\epsilon_1,\epsilon_2}} \to 1.$$

Thus there exists K such that if $L, R \geq K$, whatever is the shape of λ, μ outside these runs and $n_0 \geq R$,

$$1-\epsilon \leq \frac{\int_0^1 (1-F_{Y_\lambda}(t))(1-F_{X_\mu}(t))dt}{\int_0^1 (1-F_{Y_\lambda}(t))(1-F_{X_{\mu_{\leq n_0}}}(t))dt} \leq 1+\epsilon.$$

From now on, let us assume that the last run of λ and the first run of μ are bounded by K. Set $\eta = \inf_{\substack{\epsilon_1,\epsilon_2 \\ L,R\leq K}} B_{L,R}^{\epsilon_1,\epsilon_2}$. Since $L, R \geq 2$, thus each $B_{L,R}^{\epsilon_1,\epsilon_2}$ is strictly positive. The family $\{B_{L,R}^{\epsilon_1,\epsilon_2}\}_{\substack{\epsilon_1,\epsilon_2 \\ L,R\leq K}}$ being finite, this yields $\eta > 0$.
By Proposition 6, there exists $n_\epsilon \geq 1$ such that for $n_0 \geq n_\epsilon$ and any composition ν of size $n \geq n_0$ and first run smaller than n_0, $F_{X_{\nu_{\leq n_0}}} = F_{X_\nu} + g$ with $\|g\|_\infty \leq \epsilon\eta$.
Let $n_0 \geq n_\epsilon$ and suppose that $\lambda \vdash n, \mu \vdash m$ with the first run of μ smaller than n_0. Then there exists g such that $\|g\|_\infty \leq \epsilon\eta$, and $F_{X_{\mu_{\leq n_0}}} = F_{X_\mu} + g$. This implies

$$\begin{aligned}\int_0^1 (1-F_{Y_\lambda}(t))(1-F_{X_{\mu_{\leq n_0}}}(t))dt &= \int_0^1 (1-F_{Y_\lambda}(t))(1-F_{X_\mu}(t)-g(t))dt \\ &= \Delta - \int_0^1 g(t)(1-F_{Y_\lambda}(t))dt.\end{aligned}$$

Since $\int_0^1 g(t)(1-F_{Y_\lambda}(t))dt \leq \epsilon\eta$ and $\Delta \geq \eta$,

$$1-\epsilon \leq \frac{\int_0^1 (1-F_{Y_\lambda}(t))(1-F_{X_{\mu_{\leq n_0}}}(t))dt}{\Delta} \leq 1+\epsilon.$$

□

A corollary of the previous lemma yields a precise estimate of the probability that 1 is located in a particular valley v when the length of the slope of v goes to $+\infty$.

Corollary 2. *Let $\epsilon > 0$. There exists $n_0 \geq 1$ such that if λ is a composition, and $v \in \lambda$ is a valley with slope $\mathfrak{s}(v) = [a;b]$ of size $b-a \geq n_0$, then*

$$(1-\epsilon)\frac{b-a}{n} \leq \mathbb{P}_\lambda(1 \in v) \leq (1+\epsilon)\frac{b-a}{n}.$$

Proof. Since v is a valley, v belongs to two runs s_i, s_{i+1}. If $b-a \geq 1$, then at least one run containing v is of size larger than 2. Assume without loss of generality that $l(s_{i+1}) \geq 2$. This means that the first run of $\lambda_{>v}$ is increasing. Let $\Delta = \int_0^1 (1-F_{Y_{\lambda_{<v}}}(t))(1-F_{X_{\lambda_{>v}}}(t))dt$. Let L denote the length of the last run of $\lambda_{<v}$, and R denote the length of the first run of $\lambda_{>v}$.
If $l(s_i) = 1$, the last run of $\lambda_{<v}$ is increasing. Moreover in this case $b-a = R$. Thus the bounds on Δ from the proof of the last Lemma yield

$$\frac{1}{b-a+1}\left(1-\frac{1}{\prod_{i=2}^L (b-a+i)}\right) \leq \Delta \leq \frac{1}{b-a}\left(1-\frac{1}{\prod_{i=1}^L (b-a+i)}\right).$$

Thus independently from L, there exists n_1 such that if $l(s_i) = 1$ and $b-a \geq n_1$, then $(1-\epsilon)\frac{1}{b-a} \leq \Delta^{-1} \leq (1+\epsilon)\frac{1}{b-a}$.

If $l(s_i) > 1$, the last run of $\lambda_{<v}$ is decreasing. Then $b-a = L+R-1$, and the bounds on Δ from the proof of the previous Lemma yield

$$\frac{1}{b-a+2} \leq \Delta \leq \frac{1}{b-a}.$$

There exists n_2 such that if $l(s_i) > 1$ and $b-a \geq n_2$, then $(1-\epsilon)\frac{1}{b-a} \leq \Delta^{-1} \leq (1+\epsilon)\frac{1}{b-a}$.

Set $n_0 = \max(n_1, n_2)$, and let $n \geq n_0$. From Corollary1 $\mathbb{P}_\lambda(1 \in v) = \frac{1}{n\Delta}$, and thus

$$(1-\epsilon)\frac{b-a}{n} \leq \mathbb{P}_\lambda(1 \in v) \leq (1+\epsilon)\frac{b-a}{n}.$$

□

From the bounds $\{A^{\epsilon_1,\epsilon_2}_{R,L}, B^{\epsilon_1,\epsilon_2}_{R,L}\}$ on Δ that were found in the proof of Lemma 3, it is also possible to deduce a bound on the location probability of 1 in σ_λ:

Lemme 4. *Let λ be a composition of n, and $a < b$ be two peaks of λ. Then*

$$\mathbb{P}_\lambda(1 \in \lambda_{>a,<b}) \leq 2\frac{b-a}{n}.$$

Proof. Since 1 has to be located in a valley of λ,

$$\mathbb{P}(1 \in \lambda_{>a,<b}) = \sum_{\substack{v\in V\\ a<v<b}} \mathbb{P}(1 \in v).$$

From Corollary 1, for each $v \in V$,

$$\mathbb{P}(1 \in v) = \frac{1}{|\lambda|}\frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt},$$

Suppose that $v \in s_i \cap s_{i+1}$. By the bounds on Δ from the proof of Lemma 3,

$$\frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt} \leq l(s_i) + l(s_{i+1}) + 1 \leq 2(l(s_i) + l(s_{i+1})).$$

The latter inequality yields

$$\mathbb{P}(1 \in \lambda_{>a,<b}) \leq \frac{2}{|\lambda|}\sum_{\substack{v\in V\\ a<v<b}} l(s_i) + l(s_{i+1}) \leq \frac{2(b-a)}{n}.$$

□

7.2. **Convergence to a uniform distribution.** Let us show the convergence in law of ξ_1^λ. Let π denote the Levy-Prokhorov metric on the set $\mathcal{M}_1[0,1]$ of probability measures on $[0,1]$.

Proposition 7. *Let $\epsilon > 0$. There exists n_0 such that for $n \geq n_0$, $\lambda \vdash n$,*

$$\pi(\xi_1^\lambda, \mathcal{U}([0,1])) \leq \epsilon.$$

Proof. Let $\epsilon > 0$. Since $F_{\mathcal{U}([0,1])} = Id_{[0,1]}$ is continuous, it is enough to prove that for $s \in [0,1]$ and for λ large enough,

$$|\mathbb{P}(\xi_1^\lambda \in [0,s]) - s| \leq \epsilon.$$

Let $0 < s < 1$ and n_ϵ be the constant given by Lemma 3 for ϵ. Let $\lambda \vdash n$ and let v_n denote the last valley such that the associated slope intersects $[0, ns]$, namely $\mathfrak{s}(v^n) \cap [0, ns] \neq \emptyset$: since $0 < s$, such v_n always exists for n large enough. Let $[a_n; b_n]$ denote the slope of v_n. Thus $a_n \leq ns < b_n + 1$.
If $1 \in \lambda_{<a_n}$, $\xi_1^\lambda \in [0, \frac{a_n}{n}[\subset [0,s]$. Moreover

$$\begin{aligned}\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) &= \sum_{\substack{v \in V\\ v < a_n}} \mathbb{P}_\lambda(1 \in v)\\ &= \sum_{\substack{v \in V\\ v < a_n}} \frac{1}{|\lambda|} \frac{1}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v}}}(t))dt}.\end{aligned}$$

If $v < a_n$ is a valley, there is necessarily a peak between v and a_n, and thus the first run of $\lambda_{>v}$ is of size smaller than $v_n - v$. Therefore from Lemma 3,

$$\begin{aligned}\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq &\sum_{\substack{v \in V\\ v < a_n}} \frac{1}{n} \frac{1+\epsilon}{\int_0^1 (1 - F_{Y_{\lambda_{<v}}}(t))(1 - F_{X_{\lambda_{>v,<v_n+n_\epsilon}}}(t))dt}\\ \leq &(1+\epsilon)\frac{v_n + n_\epsilon}{n} \mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{<a_n}),\end{aligned}$$

and for the same reasons,

$$\mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \geq (1-\epsilon)\frac{v_n + n_\epsilon}{n} \mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{<a_n}).$$

The proof is now divided into two complementary cases :

- λ is such that $\frac{a_n}{v_n+n_\epsilon} > 1 - \epsilon$: in this case, $\frac{v_n+n_\epsilon-a_n}{v_n+n_\epsilon} < \epsilon$. Thus from Lemma 4, $\mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{\geq a_n}) \leq 2\epsilon$. This yields

  $$1 - 2\epsilon \leq \mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{<a_n}) \leq 1,$$

  and thus

  $$(1-2\epsilon)^2 \frac{v_n + n_\epsilon}{n} \leq \mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq (1+\epsilon)\frac{v_n + n_\epsilon}{n}.$$

  The hypothesis $\frac{a_n}{v_n+n_\epsilon} > 1 - \epsilon$ yields also

  $$(1-2\epsilon)^2 \frac{a_n}{n} \leq \mathbb{P}_\lambda(1 \in \lambda_{<a_n}) \leq \frac{1+\epsilon}{1-\epsilon}\frac{a_n}{n}.$$

On the other hand by independence between σ_λ and the family $(X^1_{(p,q)})_{p,q\in\mathbb{Q}}$,

$$\begin{aligned}\mathbb{P}(1\in v_n\cap\xi_1^\lambda\in[0,s]) =&\mathbb{P}_\lambda(1\in v_n)\frac{Leb([0,s]\cap[(a_n-1)/n,b_n/n])}{(b_n-a_n+1)/n}\\ =&\mathbb{P}_\lambda(1\in v_n)(\frac{s-a_n+1}{b_n-a_n+1}\wedge 1).\end{aligned}$$

From the definition of the slope and from the bounding Lemma 4, $\mathbb{P}(1\in v\cap\xi_1^{\lambda^n}\in[0,s])\leq\frac{2(b_n-a_n)}{n}$. Thus the latter quantity doesn't become negligible for n large only if at least one of the two runs s_i or s_{i+1} surrounding v_n tends to infinity when n grows. But in this case from Corollary 2,

$$\mathbb{P}(1\in v_n)\sim_{b_n-a_n\to\infty}(b_n-a_n)/n.$$

Therefore

$$\begin{aligned}\mathbb{P}(1\in v_n\cap\xi_1^\lambda\in[0,s]) =_{n\to\infty}&(b_n-a_n)/n\frac{s-a_n/n}{b_n/n-a_n/n}+o(1)\\ =_{n\to+\infty}&Leb([a_n/n,s]+o(1),\end{aligned}$$

with $o(1)$ being a quantity converging to zero with n, independently of the shape of λ. Summing the probabilities yields for n large enough

$$\begin{aligned}\mathbb{P}(\xi_1^\lambda\in[0,s])\leq&\frac{1+\epsilon}{1-\epsilon}Leb([0,\frac{a_n}{n}])+Leb([\frac{a_n}{n},s])+o(1)\\ \leq&\frac{1+\epsilon}{1-\epsilon}Leb([0,s])+o(1),\end{aligned}$$

and for the same reasons

$$(1-2\epsilon)^2Leb([0,s])+o(1)\leq\mathbb{P}(\xi_1^\lambda\in[0,s]).$$

There exists thus n_1 such that for $n\geq n_1$, if $\frac{a_n}{v_n+n_\epsilon}>1-\epsilon$,

$$(1-2\epsilon)^2Leb([0,s])-\epsilon\leq\mathbb{P}(\xi_1^\lambda\in[0,s])\leq\frac{1+\epsilon}{1-\epsilon}Leb([0,s])+\epsilon.$$

- λ is such that $\frac{a_n}{v_n+n_\epsilon}\leq 1-\epsilon$: this implies that either v_n remains bounded or v_n-a_n goes to $+\infty$ as n grows.
  Suppose that v_n remains bounded by K as n grows. In this case by Lemma 4, $\mathbb{P}(1\in\lambda_{<v_n})\to 0$. Thus
  $$\mathbb{P}(\xi_1^\lambda\in[0,s])=\mathbb{P}(1\in v\cap\xi_1^\lambda\in[0,s])+o(1).$$
  Since v_n remains bounded by K and $b_n+1>ns$, the slope of v_n tends to $+\infty$, and therefore from Corollary 2,
  $$\mathbb{P}(\xi_1^\lambda\in[0,s])=\frac{b_n-a_n}{n}\frac{ns-a_n}{b_n-a_n}=s+o(1).$$
  Suppose that v_n-a_n goes to $+\infty$. From Corollary 2,
  $$\mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1\in v_n)=_{n\to\infty}\frac{v_n+n_\epsilon-a_n}{v_n+n_\epsilon}+o(1).$$

From Lemma 4, $\mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{>v_n,<v_n+n_\epsilon}) \leq \frac{2n_\epsilon}{v_n+n_\epsilon} = o(1)$ and thus

$$\begin{aligned}\mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{<a_n}) =& 1 - \mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in v_n) - \mathbb{P}_{\lambda_{<v_n+n_\epsilon}}(1 \in \lambda_{>v_n,<v_n+n_\epsilon}) \\ =& \frac{a_n}{v_n+n_\epsilon} + o(1).\end{aligned}$$

Thus

$$\mathbb{P}(\xi_1^\lambda \in [0,s]) \leq (1+\epsilon)\frac{a_n}{n} + \frac{b_n - a_n}{n}\frac{ns - a_n}{b_n - a_n} + o(1) \leq (1+\epsilon)s + o(1),$$

and for the same reasons $(1-\epsilon)s + o(1) \leq \mathbb{P}(\xi_1^\lambda \in [0,s])$. This yields the existence of n_2 such that if $n \geq n_2$ and $\frac{a_n}{v_n+n_\epsilon} \leq 1-\epsilon$,

$$(1-\epsilon)Leb([0,s]) - \epsilon \leq \mathbb{P}(\xi_1^\lambda \in [0,s]) \leq (1+\epsilon)Leb([0,s]) + \epsilon.$$

By the results from both cases, there exists n_0 such that for $n \geq n_0, \lambda \vdash n$,

$$|\mathbb{P}(\xi_1^\lambda \in [0,s]) - s| \leq \epsilon.$$

□

8. Martin boundary of $\mathcal{Z}$

This section is devoted to the proof of Conjecture 1, yielding the identification of the Martin boundary of $\mathcal{Z}$ with its minimal boundary.

8.1. **Generalization of Proposition 7.**

The result of the previous section can be generalized for $k \geq 2$:

Proposition 8. *Let λ_n be a sequence of compositions of size tending to infinity. Then for $k \geq 1$,*

$$(\xi_i^{\lambda_n})_{1\leq i\leq k} \rightarrow_{law} (X_1, \dots, X_k),$$

with $(X_1, \dots, X_k)$ a vector of k independent uniform random variables on $[0,1]$.

Proof. Let us prove by recurrence on $k \geq 1$ that for $\epsilon > 0$, there exists $n_k \in \mathbb{N}$ such that for $n \geq n_k$, $\lambda \vdash n$,

$$\pi((\xi_i^\lambda)_{1\leq i\leq k}, (X_1, \dots, X_k)) \leq \epsilon,$$

π denoting the Levy-Prokhorov metric on $[0,1]^k$.
The initialization of the recurrence is done by Proposition 7. Let $k \geq 2$. It suffices to show that the law ξ_k^λ conditioned on $(\xi_i^\lambda)_{1\leq i\leq k-1}$ is close to the uniform law on $[0,1]$ when n becomes large.
Let $s \in [0,1] \setminus \mathbb{Q}, \epsilon > 0$. Let

$$\Upsilon_\eta = \cap_{1\leq i\leq k-1}\{(x_1, \dots, x_{k-1}) \in [0,1]^{k-1} | x_i \notin [s-\eta, s+\eta]\}.$$

For all η, $Leb(\partial\Upsilon_\eta) = 0$ and $Leb(\lim_{\eta\to 0}\Upsilon_\eta) = 1$, thus by the recurrence hypothesis and the portemanteau theorem, there exists $\eta > 0$ such that for λ large enough,

$$\mathbb{P}((\xi_i^\lambda)_{1\leq i\leq k-1} \in \Upsilon_\eta) \geq 1-\epsilon, \tag{7}$$

and

$$\pi(((\xi_i^\lambda)_{1\leq i\leq k-1}|A_\eta),((X_i)_{1\leq i\leq k-1}|B_\eta))\leq\epsilon, \tag{8}$$

with $A_\eta=\{(\xi_i^\lambda)_{1\leq i\leq k-1}\in\Upsilon_\eta\}$ and $B_\eta=\{(X_i)_{1\leq i\leq k-1}\in\Upsilon_\eta\}$.
Let $\lambda\vdash n$ and $\vec{i}=(i_1,\dots,i_{k-1})$ such that $\mathbb{P}(\sigma_\lambda^{-1}(1)=i_1,\dots,\sigma_\lambda^{-1}(k-1)=i_{k-1})\neq 0$. Let us further assume that $\vec{i}$ satisfies the following condition :

$$\forall 1\leq j\leq k-1,\mathfrak{s}(i_j)\not\subset[n(s-\eta),n(s+\eta)]. \tag{$*$}$$

Then λ can be decomposed as

$$\lambda=\lambda_1-\mu_1+\lambda_2-\cdots-\mu_r+\lambda_{r+1},$$

with μ_i consisting only in cells included in $\vec{i}$. From the latter construction, each run of λ intersects at most one λ_i.
Conditioned on $\mathcal{X}_{\vec{i}}=\{\sigma_\lambda^{-1}(1)=i_1,\dots,\sigma_\lambda^{-1}(k-1)=i_{k-1}\}$, the random filling of λ consists in sampling a uniformly random multiset $\vec{R}=(R_1,\dots,R_{r+1})$ of cardinal $(|\lambda_1|,\dots,|\lambda_{r+1}|)$ among $[k;n]$, and then independently filling each subcomposition $\lambda_1,\dots,\lambda_{r+1}$ respectively with $R_1,\dots,R_{r+1}$. Since k is the lowest element of $[k;n]$, for $v\in\lambda_i$, $\mathbb{P}(k\in v|\mathcal{X}_{\vec{i}})\neq 0$ if and only if v is a valley of λ_i, and if this is the case,

$$\mathbb{P}(k\in v|\mathcal{X}_{\vec{i}})=\mathbb{P}_{(R_1,\dots,R_{r+1})}(k\in R_i)\mathbb{P}_{\lambda_i}(1\in v).$$

Let $s_p=[a_p;a_{p+1}]$ be the run of λ such that $s\in[\frac{a_p-1}{n};\frac{a_{p+1}}{n}]$. If i_j is a peak, necessarily the two runs overlapping on i_j contain only elements lower than j, and thus the slope of i_j is smaller than j. Thus since for all $1\leq j\leq k-1$, $\mathfrak{s}(i_j)\not\subset[n(s-\eta),n(s+\eta)]$, for n large enough, the peak of s_p cannot be in any μ_i for $1\leq i\leq r$. This yields the existence of a unique i_0 such that $s_p\cap\lambda_{i_0}\neq\emptyset$.
Set $\lambda_{i_0}=\lambda_{>a,<b}$ and define the border B of λ_{i_0} as the set of valleys v of λ_{i_0} such that the slope $\mathfrak{s}_\lambda(v)$ of v in λ is not included in λ_{i_0}. B has at most two elements. If v is a valley of λ_{i_0} which does not belong to B,

$$\mathbb{P}_\lambda(\{1\in v\}\cap\{\xi_k^\lambda\leq s\}|\mathcal{X}_{\vec{i}})=\mathbb{P}_{\vec{R}}(k\in R_{i_0})\mathbb{P}_{\lambda_{i_0}}(\{1\in v\}\cap\{\xi_1^{\lambda_{i_0}}\in[0,\frac{ns-a}{b-a}]\}).$$

If $v\in B$, $\#(\mathfrak{s}_\lambda(v)\setminus\mathfrak{s}_{\lambda_{i_0}}(v))<k-1$. Therefore when n goes to $+\infty$, $|\mathbb{P}(X_{\mathfrak{s}_\lambda(v)}\leq s)-\mathbb{P}(X_{\mathfrak{s}_{\lambda(i_0)}(v)}\leq s)|=o(1)$, with $o(1)$ being a quantity going to 0 when n grows, independently of λ or $\vec{i}$. Thus if $v\in B$,

$$\mathbb{P}_\lambda(\{1\in v\}\cap\{\xi_k^\lambda\leq s\}|\mathcal{X}_{\vec{i}})=\mathbb{P}_{\vec{R}}(k\in R_{i_0})(\mathbb{P}_{\lambda_{i_0}}(\{1\in v\})\cap\{\xi_1^{\lambda_{i_0}}\in[0,\frac{ns-a}{b-a}])+o(1)\}).$$

Summing the probabilities yields

$$\mathbb{P}(\{\xi_k^\lambda\in[0,s]\}\cap\{k\in\lambda_{i_0}\}|\mathcal{X}_{\vec{i}})=\mathbb{P}_{\vec{R}}(k\in R_{i_0})\mathbb{P}_{\lambda_{i_0}}(\xi_k^\lambda\leq\frac{ns-a}{b-a})+o(1).$$

If $i<i_0$, $k\in\lambda_i$ implies that $\xi_k^\lambda\in[0,s]$. A standard counting argument shows that

$$\mathbb{P}_{\vec{R}}(k\in R_i)=\frac{R_i}{\sum_j R_j},$$

and thus,

$$\begin{aligned}
\mathbb{P}(\xi_k^\lambda \in [0,s] | \mathcal{X}_{\vec{i}}) &= \left(\sum_{i<i_0} \mathbb{P}_{\vec{R}}(k \in R_i) + \mathbb{P}_{\vec{R}}(k \in R_{i_0}) \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) \right) + o(1) \\
&= \left(\sum_{i<i_0} \frac{R_i}{n} + \frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) \right) + o(1) \\
&= \frac{a}{n} + \frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{R_{i_0}}]) + o(1).
\end{aligned}$$

Either R_{i_0} remains bounded as $n \to \infty$ and $\frac{R_{i_0}}{n} \mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]) \to 0$, either R_{i_0} goes to infinity, and by Proposition 7,

$$\mathbb{P}_{\lambda_{i_0}}(\xi_1^{\lambda_{i_0}} \in [0, \frac{ns-a}{b-a}]) = \frac{ns-a}{R_{i_0}} + o(1).$$

Thus in any case,

$$\mathbb{P}(\xi_k^\lambda \in [0,s] | \mathcal{X}_{\vec{i}}) = \frac{a}{n} + \frac{R_{i_0}}{n} \frac{ns-a}{R_{i_0}} + o(1) \to s,$$

and the convergence is uniform in $\lambda, \vec{i}$.
Let $(x_i)_{1 \le i \le k-1} \in \Upsilon_\eta$. If $(\xi_i^\lambda)_{1 \le i \le k-1} = (x_i)_{1 \le i \le k-1}$, then $(\sigma_\lambda^{-1}(1), \dots, \sigma_\lambda^{-1}(k-1))$ verifies the condition $(*)$. Moreover from Lemma 2, $(\xi_1^\lambda, \dots, \xi_{k-1}^\lambda) \mapsto (\sigma_\lambda^{-1}(1), \dots, \sigma_\lambda^{-1}(k-1))$ is well-defined and

$$(\xi_k^\lambda | (\xi_i^\lambda)_{1 \le i \le k-1}) = (\xi_k^\lambda | \sigma_\lambda^{-1}(\{1, \dots, k-1\}).$$

Thus for n going to $+\infty$,

$$\mathbb{P}(\xi_k^\lambda \in [0,s] | (\xi_i^\lambda)_{1 \le i \le k-1} = (x_i)_{1 \le i \le k-1}) \to s,$$

and the convergence is uniform in $(x_i)_{1 \le i \le k-1} \in \Upsilon_\eta$.
From the latter convergence and from (8), for n large enough,

$$\pi((\xi_i^\lambda)_{1 \le i \le k} | A_\eta), ((X_i)_{1 \le i \le k} | B_\eta)) \le \epsilon.$$

If ϵ is small enough, then $\mathbb{P}(A_\eta) \ge 1-\epsilon$ and $\mathbb{P}(B_\eta) \ge 1-\epsilon$ imply that

$$\pi(((\xi_i^\lambda)_{1 \le i \le k} | A_\eta), (\xi_i^\lambda)_{1 \le i \le k}) \le 2\epsilon,$$

and

$$\pi(((X_i)_{1 \le i \le k} | B_\eta), (X_i)_{1 \le i \le k}) \le 2\epsilon.$$

Thus for n large enough,

$$\pi((\xi_i^\lambda)_{1 \le i \le k}, (X_i)_{1 \le i \le k}) \le 5\epsilon.$$

This concludes the proof of the proposition. □

8.2. **Proof of Theorem 3.**

Proof. Let $(\lambda_n)_{n\geq 1}$ be a sequence of compositions and $U = (U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$ such that $\lambda_n \vdash n$ and $U_{\lambda_n} \to U$ in $\mathcal{U}^{(2)}$. By Lemma 1, $\tilde{U}_{\lambda_n} \to U$, with $\tilde{U}_\lambda$ the run paintbox defined for λ in Section 5.
Let $\mu \in \mathcal{Z}$, $\mu \vdash k$. Since $K_\mu(\lambda_n) = \frac{d(\mu,\lambda_n)}{d(\lambda_n)}$, by equality (2),

$$K_\mu(\lambda_n) = \mathbb{P}(\sigma_k^{\lambda_n} = \sigma),$$

σ being any permutation such that $des(\sigma) = \mu$. By Proposition 2,

$$\mathbb{P}(\sigma_k^{\lambda_n} = \sigma) = \mathbb{P}(\sigma_{\tilde{U}_{\lambda_n}}((\xi_i^{\lambda_n})_{1\leq i\leq k}) = \sigma).$$

By Proposition 8, as n goes to $+\infty$, $(\xi_i^{\lambda_n})_{1\leq i\leq k}$ converges in law to a sequence $(X_1, \ldots, X_k)$ of uniform independent random variables on $[0,1]$.
Thus since $\tilde{U}_{\lambda_n} \to U$, by Proposition 3

$$\sigma_{\tilde{U}_{\lambda_n}}((\xi_i^{\lambda})_{1\leq i\leq k}) \to_{law} \sigma_U((X_1, \ldots, X_k)) = \sigma_U.$$

Therefore

$$K_\mu(\lambda_n) = \mathbb{P}(\sigma_{\lambda_n} = \sigma) \to p_{(U_\uparrow, U_\downarrow)}(\mu).$$

□

As explained in Section 4, the latter Theorem implies Conjecture 1.

Corollary 3. *Conjecture 1 is true and for the graded graph $\mathcal{Z}$,*

$$\partial_{\min}\mathcal{Z} = \partial_M \mathcal{Z}.$$

We end this section by showing that the topology on $\hat{\mathcal{Z}} = \mathcal{Z} \cup \partial_M\mathcal{Z}$, abstractly constructed in Section 2.1, can be concretely described in terms of oriented Paintbox construction. From the work of Gnedin and Olshanski in [8] (Proposition 36), $\partial_{\min}\mathcal{Z}$ with the induced topology of Section 2.1 is homeomorphic to $\mathcal{U}^{(2)}$. Since from the latter Corollary, $\partial_{\min}\mathcal{Z} = \partial_M\mathcal{Z}$, as topological spaces

$$\partial_M\mathcal{Z} = \mathcal{U}^{(2)}.$$

It remains to describe the topology of $\hat{\mathcal{Z}} = \mathcal{Z} \cup \partial_M\mathcal{Z}$. Let $\mathcal{U}_n \subset \mathcal{U}^{(2)}$ be the set of $(U_\uparrow, U_\downarrow)$ such that $[0,1] \setminus U_\uparrow \cup U_\downarrow \subset \{\frac{k}{n-1}, 0 \leq k \leq n-1\}$. Then $\hat{\mathcal{Z}}$ is characterized as follows:

Corollary 4. *Let $\mathcal{T} = [0,1] \times \mathcal{U}^{(2)}$ with the product topology. As topological spaces,*

$$\hat{\mathcal{Z}} \simeq (\{0\} \times \mathcal{U}^{(2)}) \cup \bigcup_{n\geq 1} (\{\frac{1}{n}\} \times \mathcal{U}_n) \subset \mathcal{T},$$

the space on the right being considered with the induced topology from $\mathcal{T}$.

Proof. The bijection Φ is achieved by sending $\lambda \vdash n$ to $\frac{1}{n} \times U_\lambda$ and $\omega = \lim \lambda_n \in \partial_M \mathcal{Z}$ to $0 \times \lim U_{\lambda_n}$. Since $\hat{\mathcal{Z}}$ is compact, the only thing to prove is the continuity of the map.
Let $x_n \to \omega \in \hat{\mathcal{Z}}$. If $\omega \in \mathcal{Z}$, the sequence is stationary and the convergence is straightforward. Suppose that $\omega \in \partial_M \mathcal{Z}$, and divide x_n into two complementary subsequences $(x_{\phi(n)})$ and $(x_{\phi^c(n)})$ such that $x_{\phi(n)} \in \mathcal{Z}$ and $x_{\phi^c(n)} \in \partial_M \mathcal{Z}$.
By Proposition 36 of [8],

$$\Phi(x_{\phi^c(n)}) \to \Phi(\omega).$$

By Corollary 3, since $x_{\phi(n)} \to_{\hat{\mathcal{Z}}} w$,

$$\Phi(x_{\phi(n)}) = U_{x_{\phi(n)}} \to \Phi(\omega),$$

which concludes the proof. □

9. The Plancherel measure

The purpose of this section is to investigate the Plancherel measure on the graph $\mathcal{Z}$, which is the point $(\emptyset, \emptyset)$ of $\partial_M \mathcal{Z}$. We first recall the link between $\mathcal{Z}$ and the Young graph $\mathcal{Y}$ to justify the name of Plancherel measure. This link was already explained in [8] in terms of associated algebras of functions, but it seems to us that no direct link on the level of paths was clearly defined. It is the purpose of the second paragraph to clearly establish this link on the level of paths.

9.1. **The graph $\mathcal{Y}$.** A partition ρ of n is the data of a decreasing sequence on integers $(\rho_1 \geq \cdots \geq \rho_r)$ such that $\sum \rho_i = n$. n is called the degree of ρ and is denoted $\rho \vdash n$. Let us denote by $l(\rho)$ the length of the sequence. The set of partitions of n is denoted $\mathcal{Y}_n$, and the set of all partitions $\mathcal{Y}$. $\mathcal{Y}$ is ordered by saying that $\rho \preceq \tau$ if and only if $l(\rho) \leq l(\tau)$ and for all $1 \leq i \leq l(\rho)$, $\rho_i \leq \tau_i$.
As for compositions, a Young diagram is associated to each partition by drawing ρ_1 cells on the first row, ρ_2 cells on the second row and so on, such that the first cell of the row $i+1$ is just below the first cell of the row i. A standard filling of ρ is a filling of ρ with elements of $\{1, \ldots, n\}$, such that the filling is increasing to the right and to the bottom. We denote by T_ρ the set of standard fillings of ρ (also called stantard tableau of shape ρ). Here is an example of a partition $\rho = (7, 4, 2, 1)$ and a standard filling of the associated diagram.

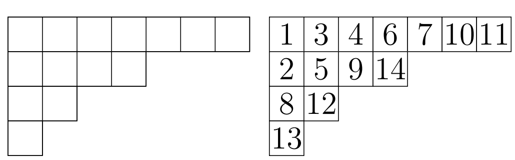

Figure 4. Young diagram of $(7, 4, 2, 1)$ and an example of standard filling

We say that $\rho \nearrow \tau$ if and only if $\deg \tau = \deg \rho + 1$ and $\rho \preceq \tau$. When $T \in T_\tau$ is a standard tableau of shape $\tau \vdash n$, $T_\downarrow$ is defined as the standard tableau obtained

by deleting the cell containing n. In particular $T_{\downarrow}$ has a shape ρ such that $\rho \nearrow \tau$. Adding an edge from ρ to τ if and only if $\rho \nearrow \tau$ transforms $\mathcal{Y}$ into a graded graph. The latter graph is a major construction for the representation theory of the groups $(\mathfrak{S}_n)_{n\geq 1}$, since the irreducible representations V_τ of $\mathfrak{S}_n$ are indexed by elements τ of $\mathcal{Y}_n$, and there is a decomposition

$$Res(V_\tau)^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} = \sum_{\rho\nearrow\tau} V_\rho.$$

As for the graph $\mathcal{Z}$, the set of paths on $\mathcal{Y}$ between the root $\emptyset$ and a partition ρ is in bijection with the set of standard tableaux of shape ρ, and each element of $\partial_{\min}\mathcal{Y}$ yields a random path on the graph $\mathcal{Y}$ (namely an infinite standard tableau). The minimal and Martin boundaries of $\mathcal{Y}$ have been intensively studied (see [12],[13],[20]) and fully described. In particuler the equality $\partial_{\min}\mathcal{Y} = \partial_M\mathcal{Y}$ holds also in this setting, and

$$\partial_M\mathcal{Y} = \{(a_1 \geq a_2 \geq \cdots \geq 0), (b_1 \geq b_2 \geq \cdots \geq 0), \sum a_j + b_j \leq 1\}.$$

For each $\omega \in \partial_M\mathcal{Y}$, ρ_ω denotes the random path on $\mathcal{Y}$ according to the harmonic measure ω.
The next paragraph establishes a link between $\mathcal{Y}$ and $\mathcal{Z}$ based on the algorithm RSK of Robinson, Schensted and Knuth. The relation between both graphs has been already established through the ring of symmetric functions and the one of quasisymmetric functions. The reader should refer to [8] for a complete review of the subject.

9.2. **RSK algorithm and the projection $\mathcal{Z} \to \mathcal{Y}$.** Let us first recall the RSK algorithm in the special case of permutations. This algorithm, initiated by Robinson in [3] and created by Schensted in [17], establishes a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of n of the same shape. Let $\sigma = (\sigma(1), \ldots, \sigma(n)) \in \mathfrak{S}_n$. The algorithm constructs a pair of standard tableaux from σ as follows :

(1) Start with an infinite array $A^0 = (a^0_{k,l})_{k,l\geq 1}$ such that each cell is filled with the entry $n+1$ (namely $a^0_{k,l} = n+1$), and an infinite array $B = (b_{k,l})_{k,l\geq 1}$ such that each cell is empty (B is called the recording tableau).
(2) At each step i, $1 \leq i \leq n$, the following insertion is done on the array A^{i-1}:
   - Let $(1, l_1)$ be the first cell (starting from the left) on the first row of A^{i-1} such that $\sigma(i) \leq a^{i-1}_{1,l_1}$. Set $a^i_{1,l_1} = \sigma(i)$.
   - Let $(2, l_2)$ be the first cell on the second row of A^{i-1} such that $a^{i-1}_{1,l_1} \leq a^{i-1}_{2,l_2}$. Set $a^i_{2,l_2} = a^{i-1}_{1,l_1}$.
   - Continue the process until the step k_0 where $a^{i-1}_{k_0,l_{k_0}} > n$. For $k > k_0$ or $k \leq k_0, l \neq l(k)$, define $a^i_{k,l} = a^{i-1}_{k,l}$. Return $A^i = (A^i_{k,l})_{k,l\geq 1}$. Set $b_{k_0,l_{k_0}} = i$.
(3) Let $P(\sigma)$ be the part of the array A^n containing entries lower or equal to n, and $Q(\sigma)$ the part of the array B consisting in non empty cells.

Then the following Theorem holds ([17], [3]):

Theorem 4. *The map $S : \sigma \mapsto (P(\sigma), Q(\sigma))$ is a bijection between $\mathfrak{S}_n$ and pairs of standard tableaux of n of the same shape. Moreover*

$$(P(\sigma^{-1}), Q(\sigma^{-1})) = (Q(\sigma), P(\sigma)).$$

From now on $\rho(\sigma)$ denotes the shape of $P(\sigma)$ (or $Q(\sigma)$).
The link between $\mathcal{Z}$ and $\mathcal{Y}$ resides in the following proposition, mapping paths on $\mathcal{Z}$ to paths on $\mathcal{Y}$.

Proposition 9. *Let $(\sigma_k)_{k\geq 1}$ be a path on $\mathcal{Z}$. Then $(\rho(\sigma_k))_{k\geq 1}$ is a path on $\mathcal{Y}$. Moreover if $\sigma = (\sigma_k)_{k\geq 1}$ is a random path on $\mathcal{Z}$, then $\rho(\sigma) = (\rho(\sigma_k))_{k\geq 1}$ is a random path on $\mathcal{Y}$ and for $P \in \mathcal{T}_\tau$ a path on $\mathcal{Y}$ between $\emptyset$ and $\tau \vdash k_0$,*

$$\mathbb{P}((\rho(\sigma_1), \dots, \rho(\sigma_{k_0})) = P) = \sum_{\substack{\sigma\in\mathfrak{S}_{k_0}\\ P(\sigma)=P}} \mathbb{P}(\sigma_{k_0} = \sigma).$$

Proof. Let $\sigma = (i_1, \dots, i_{k-1}, n, i_{k+1}, \dots) \in \mathfrak{S}_n$. If suffices to prove that

$$P(\sigma_\downarrow) = P(\sigma)_\downarrow.$$

Although the latter equality appears clearly in the algorithm, the proof is easier to write by using σ^{-1} : indeed write $\sigma^{-1} = (j_1, \dots, j_{n-1}, k)$. Since $\sigma_\downarrow = (i_1, \dots, i_{k-1}, i_{k+1}, \dots)$, $(\sigma_\downarrow)^{-1} = (j_1^*, \dots, j_{n-1}^*)$, with $j_l^* = j_l$ if $j_l < k$ and $j_l^* = j_l - 1$ if $j_l > k$. All the j_l^* (resp j_l) are distinct and thus

$$std((j_1^*, \dots, j_{n-1}^*)) = std((j_1, \dots, j_{n-1})).$$

Since the Schensted algorithm only depends on the relative values of the entries, the recording tableaux B of the algorithm for $(\sigma_\downarrow)^{-1}$ and σ^{-1} after $n-1$ steps are the same. Therefore

$$Q((\sigma_\downarrow)^{-1}) = Q(\sigma^{-1})_\downarrow.$$

Thus from Theorem 4,

$$P(\sigma_\downarrow) = Q(\sigma_\downarrow^{-1}) = Q(\sigma^{-1})_\downarrow = P(\sigma)_\downarrow.$$

This yields that $\rho(\sigma_\downarrow) \nearrow \rho(\sigma)$ and for any arrangement $(\sigma_k)_{k\geq 1}$, the sequence $(\rho(\sigma_k))_{k\geq 1}$ is a well-defined path on $\mathcal{Y}$.
In particular if $(\sigma_k)_{k\geq 1}$ is a random path on $\mathcal{Z}$ and $P \in T_\tau$, $\tau \vdash k_0$, summing the probabilities of each path yields

$$\mathbb{P}((\rho(\sigma_1), \dots, \rho(\sigma_{k_0})) = P) = \sum_{\substack{\alpha\in\mathfrak{S}_{k_0}\\ P(\alpha)=P}} \mathbb{P}_{\sigma_k}(\sigma_{k_0} = \alpha).$$

□

The important fact is that harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.

Corollary 5. *Let $\sigma = (\sigma_k)_{k\geq 1}$ be a random arrangement such that $\mathbb{P}(\sigma_k = \alpha)$ depends only on $Q(\alpha)$. Then $\rho(\sigma)$ yields an harmonic measure on $\mathcal{Y}$.*
In particular harmonic measures on $\mathcal{Z}$ yield harmonic measures on the graph $\mathcal{Y}$.

Proof. From Section 2, a random path $\rho = (\rho_k)_{k\geq 1}$ on $\mathcal{Y}$ comes from an harmonic measure if and only if for any partition $\tau \vdash n$, and $P_1, P_2 \in T_\tau$,

$$\mathbb{P}((\rho_1, \dots, \rho_n) = P_1) = \mathbb{P}((\rho_1, \dots, \rho_n) = P_2).$$

Let $\sigma = (\sigma_k)_{k\geq 1}$ be a random arrangement such that $\mathbb{P}(\sigma_k = \alpha) = p(Q(\alpha))$, with p a positive function on standard Young tableaux. From Proposition 9, for $k_0 \geq 1, \tau \vdash k_0$ and $P \in T_\tau$,

$$\begin{aligned}\mathbb{P}((\rho_1, \dots, \rho_{k_0}) = P) &= \sum_{\substack{\alpha\in\mathfrak{S}_{k_0}\\ P(\alpha)=P}} \mathbb{P}(\sigma_{k_0} = \alpha)\\ &= \sum_{\substack{\alpha\in\mathfrak{S}_{k_0}\\ P(\alpha)=P}} p(Q(\alpha)) = \sum_{Q\in T_\tau} p(Q),\end{aligned}$$

the last equality being due to Theorem 4. Thus $\mathbb{P}((\rho_1, \dots, \rho_{k_0}) = P)$ is independent of $P \in T_\tau$.
Let ϕ be an harmonic measure on $\mathcal{Z}$. From Section 3, ϕ yields a random arrangement $\sigma = (\sigma_k)_{k\geq 1}$ such that $\mathbb{P}(\sigma_k = \alpha) = p(des(\alpha))$, for a particular function $p : \mathcal{Z} \to \mathbb{R}^+$. By a standard combinatoric result (see [18]), i is a descent of α if and only if $i+1$ is in a strictly lower row than i in $Q(\alpha)$. Thus if $Q(\alpha) = Q(\alpha')$, then $des(\alpha) = des(\alpha')$ and $\mathbb{P}(\sigma_k = \alpha) = \mathbb{P}(\sigma_k = \alpha')$. From the first part of the Corollary, $\rho(\sigma)$ yields an harmonic measure on $\mathcal{Y}$. $\square$

In general, for Q a standard tableau, $des(Q)$ denotes the set of indices i such that $i+1$ is in a strictly lower row than i. This yields in particular the following equality for $\lambda \in \mathcal{Z}$:

$$d_{\mathcal{Z}}(\emptyset, \lambda) = \sum_{\tau\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset, \tau)\#\{Q \in T_\tau, des(Q) = D_\lambda). \tag{9}$$

The latter equation yields the law of $\rho(\sigma_\lambda)$, when σ_λ is chosen uniformly on the set of paths between $\emptyset$ and λ.

Lemme 5. *Let $\lambda \vdash n$ and σ_λ a uniform random path between $\emptyset$ and λ. Then $\rho(\sigma_\lambda)$ is a random path on $\mathcal{Y}$ with law*

$$\mathbb{P}(\rho(\sigma_\lambda)(k) = \tau) = d_{\mathcal{Y}}(\emptyset, \tau)K_\tau(\rho_\lambda),$$

for $\tau \in \mathcal{Y}_k$ and ρ_λ a random element of $\mathcal{Y}_n$ with law

$$\mathbb{P}(\rho_\lambda = \rho) = \mathbb{P}(Q(\sigma_\lambda) \in T_\rho).$$

Proof. Let us apply the Schensted algorithm to σ_λ. If $P \in T_\tau$, with τ a Young diagram of k cells ($k \leq n$),

$$\mathbb{P}(\rho(\sigma_\lambda)_k = \tau) = \frac{1}{d_{\mathcal{Z}}(\lambda)} \#\{\sigma \in \mathfrak{S}_n, des(\sigma) = \lambda, P(\sigma_{\downarrow k}) = \tau\}.$$

Equation (9) transforms the latter expression into

$$\begin{aligned}\mathbb{P}(\rho(\sigma_\lambda)_k = \tau) =& \frac{\sum_{\rho\in\mathcal{Y}} \#\{Q \in T_\rho, des(Q) = \lambda)\}\#\{\sigma, Q(\sigma) = Q, P(\sigma_{\downarrow k}) = \tau)\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q \in T_\rho, des(Q) = \lambda\}}\\ =&\sum_{\rho\in\mathcal{Y}} \frac{d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q \in T_\rho, des(Q) = \lambda\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q \in T_\rho, des(Q) = \lambda\}} \frac{d_{\mathcal{Y}}(\emptyset,\tau)d_{\mathcal{Y}}(\tau,\rho)}{d_{\mathcal{Y}}(\emptyset,\rho)}\\ =& d_{\mathcal{Y}}(\emptyset,\tau)K_\tau(\rho_\lambda),\end{aligned}$$

with $\rho_\lambda = \rho(\sigma_\lambda)$ a random variable on $\mathcal{Y}_n$ with law

$$\mathbb{P}(\rho_\lambda = \rho) = \frac{d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q \in T_\rho, des(Q) = \lambda\}}{\sum_{\rho\in\mathcal{Y}} d_{\mathcal{Y}}(\emptyset,\rho)\#\{Q \in T_\rho, des(Q) = \lambda\}} = \mathbb{P}(Q(\sigma_\lambda) \in T_\rho).$$

□

We finally prove that ρ restricts on $\partial_M\mathcal{Z}$ to a surjective map $p : \partial_M\mathcal{Z} \to \partial_M\mathcal{Y}$.

Proposition 10. *Let $U = (U_\uparrow, U_\downarrow) \in \partial_M\mathcal{Z}$. Let $(a_1 \geq a_2 \geq \cdots \geq 0)$ (resp. $(b_1 \geq b_2 \geq \cdots \geq 0)$) be the lengths of the interval components of $U_\uparrow$ (resp. $U_\downarrow$) in decreasing order. Then $\omega(U) = ((a_1 \geq a_2 \geq \cdots \geq 0),((b_1 \geq b_2 \geq \cdots \geq 0)) \in \partial_M\mathcal{Y}$ and $\rho(\sigma_U) = \rho_{\omega(U)}$. Moreover the induced map*

$$p : \partial_M\mathcal{Z} \to \partial_M\mathcal{Y}$$

is surjective.

Proof. Since the Schensted algorithm relates a finite number of permutations to each Young diagram, the map ρ sending random paths of $\mathcal{Z}$ to random paths of $\mathcal{Y}$ is clearly continuous with respect to the topology of convergence in law. Thus it is enough to prove the result on a dense subset of $\partial_M\mathcal{Z}$. Let $U = (U_\uparrow, U_\downarrow) \in \partial_M\mathcal{Z}$ be such that $\bar{U} = [0,1]$ and U has a finite number of interval components. Denote by $(a_1 \geq a_2 \geq \cdots \geq a_n)$ (resp. $(b_1 \geq b_2 \geq \cdots \geq b_m)$) the lengths of the interval components of $U_\uparrow$ (resp. $U_\downarrow$) in decreasing order, and $\omega_U = ((a_1 \geq a_2 \geq \cdots \geq a_n),(b_1 \geq b_2 \geq \cdots \geq b_m))$. Then σ_U can be approximated by a sequence σ_{λ_n} with $\lambda_n \vdash n$, $U(\lambda_n) \to U$. By Greene's Theorem (see [9]) and Lemma 2 of the paper [21] of Kerov and Vershik, almost surely $\rho(\sigma_{\lambda_n})$ converges in $\mathcal{Y} \cup \partial_M\mathcal{Y}$ to ω_U. Thus by identification of the Martin boundary on $\mathcal{Y}$ and Lemma 5,

$$\mathbb{P}(\rho(\sigma_{\lambda_n})(k) = \tau) \to d(\emptyset,\tau)K_{\omega_U}(\tau),$$

for $\tau \vdash k$.

In particular $\rho(\sigma_U) = \rho_{\omega(U)}$. Since the subset

$$\{U \in \mathcal{U}^{(2)}, \bar{U} = [0,1], U \text{ has a finite number of components}\}$$

is dense in $\mathcal{U}^{(2)}$, the latter equality holds on $\mathcal{U}^{(2)}$.
For every element $\omega = ((a_1 \geq a_2 \geq \cdots \geq 0), (b_1 \geq b_2 \geq \cdots \geq 0)) \in \partial_M \mathcal{Y}$, it is possible to find $U \in \mathcal{U}^{(2)}$ such that $\omega_U = \omega$, thus the map

$$p : \begin{cases} \partial_M \mathcal{Z} & \longrightarrow & \partial_M \mathcal{Y} \\ U & \mapsto & \omega_U \end{cases}$$

is surjective. $\square$

9.3. **Asymptotic of λ_n under the Plancherel measure.** The purpose of this subsection is to explore further the behavior of $\sigma_{(\emptyset,\emptyset)}$. $\sigma_{(\emptyset,\emptyset)}$ is called the Plancherel measure on $\mathcal{Z}$ since it is the only element of $\partial_M \mathcal{Z}$ that yields the Plancherel measure on $\mathcal{Y}$ through the map p of the last paragraph.
In order to describe the descent set of a permutation $\sigma \in \mathfrak{S}_{n+1}$ we introduce the following notations. Let f_σ be the piecewise linear function such that $f_\sigma(0) = 0$, and

$$f_\sigma(i) - f(i-1) = \begin{cases} -1 & \text{if } i \text{ is a descent of } \sigma \\ +1 & \text{othewise} \end{cases}$$

To describe the asymptotic value of f_σ for σ following the probability measure $p_{(U_\uparrow, U_\downarrow)}$ ($(U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$), we define also the following function $f_{(U_\uparrow, U_\downarrow)}$: it is the unique function such that

$$\begin{cases} f_{(U_\uparrow,U_\downarrow)}(0) = 0 & \\ f'_{(U_\uparrow,U_\downarrow)}(t) = 1 & \text{if } t \in U_\uparrow \\ f'_{(U_\uparrow,U_\downarrow)}(t) = -1 & \text{if } t \in U_\downarrow \\ f'_{(U_\uparrow,U_\downarrow)}(t) = 0 & \text{if } t \in [0,1] \setminus U \end{cases}.$$

The map $(U_\uparrow, U_\downarrow) \mapsto f_{(U_\uparrow,U_\downarrow)}$ is continuous from $\mathcal{U}^{(2)}$ to $\mathcal{C}([0,1], \mathbb{R})$, and the following result holds :

Proposition 11. *Let $U \in \mathcal{U}^{(2)}$. Then*

$$(t \mapsto \frac{1}{n} f_{\sigma_U(n)}(nt)) \to_{p.s, \|\cdot\|_\infty} f_{(U_\uparrow,U_\downarrow)}.$$

The proof is a deduction from Theorem 1, since $U(\sigma_U(n)) \to_{\mathcal{U}^{(2)}} U$.
The next step is to get the fluctuations of f_{σ_U}. Only the case $U = (\emptyset, \emptyset)$ is done here. The result consists mainly in a mathematical formalization of the results obtained by Oshanin and Voituriez from a physical point of view in [16]. The reader should refer to the latter paper for interesting additional informations on the process $f_{\sigma_{\emptyset,\emptyset}}$.

Theorem 5. *For σ_n being uniformly sampled among $\mathfrak{S}_n$,*

$$(t \mapsto \frac{1}{\sqrt{n}} f_{\sigma_n}(nt)) \to \frac{1}{\sqrt{3}} \mathcal{B},$$

$\mathcal{B}$ denoting the Brownian motion on $[0,1]$.

Proof. Recall from Section 4 that σ_n can be sampled from a family of independent uniform random variables $(x_i)_{i\geq 1}$ on $[0,1]$ by applying the map std^{-1} on the sequence $(x_i)_{1\leq i\leq n}$. Since $\sigma \mapsto \sigma^{-1}$ is a measure preserving map (uniquely for the uniform measure), $f_{\sigma_n} \sim f_{\sigma_n^{-1}}$. The property noticed by Oshanin and Voituriez is that $((f_{\sigma_n^{-1}}, x_n))_{n\geq 1}$ is a Markov chain : indeed i is a descent of σ_n^{-1} if and only if $x_i > x_{i+1}$. Therefore, $Des(\sigma_{n+1}^{-1}) \cap \{1, \dots, n-1\} = Des(\sigma_n^{-1})$, and $n \in Des(\sigma_{n+1}^{-1})$ if and only if $x_n > x_{n+1}$. In the sequel $f_{\sigma_n^{-1}}(i)$ is denoted by Y_i (the subscript n is dropped, since this depends only on $(\sigma_n)_{\downarrow i}$).
This yields that for $R = [r_1; r_1 + r_2]$ and $S = [s_1; s_1 + s_2]$, with $s_1 \geq r_1 + r_2 + 2$, $n \geq s_1 + s_2$, we have

$$(\#Des(\sigma_n) \cap R, \#Des(\sigma_n) \cap S)) \sim X_1 \otimes X_2,$$

with $X_1 \sim \#Des(\sigma_{r_2+1})$ and $X_2 \sim \#Des(\sigma_{s_2+1})$. Moreover the number of permutations of n with k descents is given by the Eulerian number A_k^n, and its asymptotic value (see [18]) gives:

Lemme 6. *For n going to infinity, and σ_n uniform on $\mathfrak{S}_n$,*

$$\frac{1}{\sqrt{n}}(\#Des - \frac{n}{2}) \to_{law} \mathcal{N}(0, 1/12).$$

The latter Lemma together with the strong Markov property shows that if we write $\tilde{f}_{\sigma_n}(t) = \frac{1}{\sqrt{n}} f_{\sigma_n}(nt)$, the marginal distributions of $\tilde{f}_{\sigma_n}$ converge towards the ones of $\frac{1}{\sqrt{3}}\mathcal{B}$. An adequate bound for $\|f_{\sigma_n}\|$ is needed to be able to conclude by stantard tightness arguments. We follow the Theorem 8.4 of the book [2] of Billingsley :

Theorem 6. *Let $(Y_i)_{i\geq 0}$ be a real random process. Let $f_n : [0,1] \to \mathbb{R}$ define the linear interpolation between the points*

$$f_n(\frac{i}{n}) = \frac{1}{\sqrt{n}} Y_i.$$

Suppose that for all $\epsilon > 0$, there exists $\lambda > 0, n_0 \geq 0$ such that for all $k \in \mathbb{N}, n \geq n_0$,

$$\mathbb{P}(\max_{i\leq n} |Y_{k+i} - Y_k| \geq \lambda\sigma\sqrt{n}) \leq \epsilon/\lambda^2.$$

Then the sequence f_n is tight.

The hypothesis of the Theorem is verified through the following Lemma, that mimicks the situation coming from a usual random walk.

Lemme 7. *Let $S_n = \sup_{[0,n]} Y_n, a > 0$ and $b \leq a - 2$. Then*

$$\mathbb{P}(S_n \geq a, Y_n \leq b) \leq \mathbb{P}(Y_n \geq 2a - b - 2),$$

and $F_{S_n}(t) \geq F_{|Y_n|}(t)$ for all $t \in \mathbb{R}$.

Proof. In the Markov chain (Y_n, x_n), $T = \inf(u \in \mathbb{N}, Y_u = a)$ is a stopping time. Since $\{S_n \geq a\} = \{T \leq n\}$, $\{S_n \geq a\} \in \mathcal{F}_T$ and by the strong Markov property,

$$\begin{aligned}\mathbb{P}(S_n \geq a, Y_n \leq b) =& \mathbb{P}((T \leq n) \cap (Y_n - Y_T \leq b - a))\\ =& \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_0 \leq b - a))\\ \leq& \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)).\end{aligned}$$

Since $\tilde{Y}_{n-T} - \tilde{Y}_1$ is independent of the value $\tilde{Y}_0 = Y_T$,

$$\mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(Y_T, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)) = \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} - \tilde{Y}_1 \leq b - a + 1)).$$

Moreover $\tilde{Y}_{n-T} - \tilde{Y}_1 \sim -(\tilde{Y}_{n-T} - \tilde{Y}_1)$, thus

$$\begin{aligned}\mathbb{P}(S_n \geq a, Y_n \leq b) \leq& \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(0, x_T)}(-(\tilde{Y}_{n-T} - \tilde{Y}_1) \leq b - a + 1))\\ =& \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} \geq a - (b+1) + \tilde{Y}_1))\\ \leq& \mathbb{E}(\mathbf{1}_{T\leq n}\mathbb{P}_{(0, x_T)}(\tilde{Y}_{n-T} \geq a - (b+1) - 1))\\ \leq& \mathbb{P}((T \leq n) \cap (Y_n \geq 2a - b - 2)) \leq \mathbb{P}(Y_n \geq 2a - b - 2),\end{aligned}$$

the last equality being due to the fact that $(Y_n \geq 2a - b - 2) \subset (T \leq n)$. This yields

$$\begin{aligned}\mathbb{P}(S_n \geq a) \leq& \mathbb{P}((S_n \geq a) \cap (Y_n \leq a - 2)) + \mathbb{P}((S_n \geq a) \cap (Y_n \geq a))\\ \leq& \mathbb{P}(Y_n \geq a) + \mathbb{P}(Y_n \geq a) \leq \mathbb{P}(|Y_n| \geq a),\end{aligned}$$

the last equality being due to the fact that the law of Y_n is symmetric. This yields

$$F_{S_n}(u) \geq F_{|Y_n|}(u).$$

□

In particular from the latter Lemma, for $\epsilon > 0$ and λ such that $\mathbb{P}(\mathcal{N}(0, 1/3) \geq \lambda) \leq \frac{\epsilon}{2\lambda^2}$, and for n large enough, $k \geq 0$,

$$\begin{aligned}\mathbb{P}(\max_{i\leq n} |Y_{k+i-Y_k}| \geq \lambda\sqrt{n}) \leq& \mathbb{P}(|Y_{k+n} - Y_k| \geq \lambda\sqrt{n})\\ \leq& \mathbb{P}(\mathcal{N}(0, 1/3) \geq \lambda) + \frac{\epsilon}{2\lambda^2}\\ \leq& \frac{\epsilon}{\lambda^2}.\end{aligned}$$

And this concludes the proof of the proposition. □

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Appendix: convergence result for the Paintbox construction

This appendix is dedicated to the proof of Proposition 3. Some notations and two preliminary results are first given.

9.4. **Cluster sets.** Let $k \geq 1$ and A a given set. We define an $A-$cluster of k as a map $f : A \to \mathcal{P}([1;k])$ such that $f(a_1) \cap f(a_2) = \emptyset$ for $a_1 \neq a_2$. The residue of f is the set $R_f = [1;k] \setminus \bigcup f(a)$ and the support of f is the set S_f of $a \in A$ such that $f(a) \neq \emptyset$. The degree of f is the minimum of the cardinals of non-empty sets $f(a)$.The set of $A-$clusters (resp. $A-$clusters of degree larger than s) is denoted $C^k(A)$ (resp $C_s^k(A)$). For $1 \leq s \leq k$, the $s-$level of the $A-$cluster f, denoted f^s, is the $A-$cluster of $C_s^k(A)$ defined by :

$$f^s(a) = \begin{cases} f(a) & \text{if } |f(a)| \geq s \\ \emptyset & \text{else} \end{cases}.$$

Let $\vec{x} = (x_1, \dots, x_k)$ be a sequence on a space $(\Omega^k, \mathcal{A}^{\otimes k})$. Then any set J and any collection of disjoint subsets $\mathcal{A} = (\mathcal{A}_j)_{j \in J}$ of Ω yields a $J-$cluster map of k

$$f_{\vec{x},\mathcal{A}}(j) = \{i | x_i \in \mathcal{A}_j\}.$$

For $U \in \mathcal{U}^{(2)}$, denote by $\mathcal{U} = \{U_\alpha\}_{\alpha \in A_U}$ the collection of interval components of $U_\uparrow \cup U_\downarrow$. We say that $\alpha \in A_U^+$ (resp. A_U^-) if $U_\alpha \subset U_\uparrow$ (resp. $U_\alpha \subset U_\downarrow$).

Lemme 8. *Let $U = (U_\uparrow, U_\downarrow) \in \mathcal{U}^{(2)}$. For $\sigma \in \mathfrak{S}_k$ and $f \in C_2^k(A_U)$ define the sets*

$$X_{\sigma,f}(U) = \{\vec{x} \in [0,1]^k | std^{-1}(\vec{x}) = \sigma\} \cap \{(f_{\vec{x},\mathcal{U}})^2 = f\}.$$

Then the sets $X_{\sigma,f}(U)$ are disjoint open sets and σ_U is constant on each of these sets. In particular if $X_k = (X(i))_{1 \leq i \leq k}$ is a random variable with density on $[0,1]^k$, $\sigma_U(X_1, \dots, X_k)$ is $((X_{\sigma,f}(U))_{f,\sigma}$ measurable.

Proof. Let us write simply $X_{\sigma,f}$ instead of $X_{\sigma,f}(U)$. Suppose that $X_k \in X_{\sigma,f} \cap X_{\sigma',f'}$. Then $std^{-1}(X_k) = \sigma = \sigma'$. Moreover $f = (f_{X_k,\mathcal{U}})^2 = f'$ and $X_{\sigma,f} = X_{\sigma',f'}$. The events $X_{\sigma,f}$ are thus disjoint for distinct pairs (σ, f). They are open from their definition and the fact that $U_\uparrow, U_\downarrow$ are open sets.
Each $\tau \in \mathfrak{S}_k$ is entirely defined by the set $S_\tau = \{(i,j) | i < j, \tau^{-1}(i) \leq \tau^{-1}(j)\}$. Let $\sigma \in \mathfrak{S}_k$. Then from the Paintbox construction, $\sigma_U^{-1}(\{\tau\}) \cap \{X_k | std^{-1}(X_k) = \sigma\}$ is precisely the set of X_k such that by writing $f = f_{X_k,\mathcal{U}}$:

- $(i,j) \in S_\sigma \cap S_\tau \Rightarrow \forall a \in A_U^-, \{i,j\} \not\subset f(a)$,
- $(i,j) \in S_\sigma \setminus S_\tau \Rightarrow \exists a \in A_U^-, \{i,j\} \subset f(a)$,
- $(i,j) \in S_\tau \setminus S_\sigma \Rightarrow \exists a \in A_U^+, \{i,j\} \subset f(a)$,
- $(i,j) \notin S_\sigma \cup S_\tau \Rightarrow \forall a \in A_U^+, \{i,j\} \not\subset f(a)$.

Define by $D(\sigma, \tau)$ the set of $A(U)-$clusters that respect the above four conditions. Thus

$$\sigma_U^{-1}(\{\tau\}) \cap \{X_k | std^{-1}(X_k) = \sigma\} = \bigcup_{f \in D(\sigma,\tau)} X_{\sigma,f}.$$

Since X_k admits a density function, $[0,1]^k \setminus \bigcup\{X_k | std^{-1}(X_k) = \sigma\}$ is a null-set and thus :

$$\sigma_U^{-1}(\{\tau\}) = \bigcup_{\sigma} \bigcup_{f \in D(\sigma,\tau)} X_{\sigma,f},$$

which proves the Lemma. □

9.5. **Convergence in law with conditioning.** The pattern of the proof of Proposition 3 implies the following question: suppose that A, B are metric spaces, considered as measure spaces with a given measure on each associated borelian $\sigma-$algebra. Let $f_n : A \to B$ be a sequence of measurable functions that converges pointwise almost surely to a continuous function $f : A \to B$. Let $(X_n)_{n\geq 1}$ be a sequence of random variables on A that converges in law to a random variable X. Do we have the convergence in law $f_n(X_n) \to f(X)$? The answer is negative in general, but in a very particular case the result holds.

Lemme 9. *Let $(\mathcal{X}_m)_{m\geq 1}$ be a family of measurable spaces of A with the following conditions :*

- $\lim\limits_{m\to\infty} \mathbb{P}(X \in \mathcal{X}_m) = 1.$
- $\forall m \geq 1, \mathbb{P}(X \in \partial\mathcal{X}_m) = 0.$
- *For all $m \geq 1$, $f_{n|\mathcal{X}_m} \to f_{|\mathcal{X}_m}$ uniformly.*

Then $f_n(X_n)$ converges in law to $f(X)$.

Proof. Let $g : B \to \mathbb{R}$ be a $1-$Lipschitz function bounded by 1. It suffices to show that $\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X)) \longrightarrow 0$. For each $m \geq 1$, the difference can be bounded by

$$\begin{aligned}
&|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X))| \leq \\
&\quad |\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m)| + |\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X_n)|\mathcal{X}_m)| \\
&\quad + |\mathbb{E}(g \circ f(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X)|\mathcal{X}_m)| + |\mathbb{E}(g \circ f(X)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X))|.
\end{aligned}$$

Let m be such that $\mathbb{P}(X \in \mathcal{X}_m) \geq 1 - \epsilon$. Since $\mathbb{P}(X \in \partial\mathcal{X}_m) = 0$, by the convergence in law there exists n_0 such that for $n \geq n_0$, $\mathbb{P}(X_n \in \mathcal{X}_m) \geq 1 - 2\epsilon$. For $n \geq n_0$,

$$\mathbb{E}(g \circ f_n(X_n)) = \mathbb{P}(X_n \in \mathcal{X}_m)\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) + \mathbb{P}(X_n \notin \mathcal{X}_m)\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m^c).$$

Since g is bounded by 1 and $\mathbb{P}(X_n \notin \mathcal{X}_m) \leq 2\epsilon$,

$$\begin{aligned}
|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m)| \leq& 2\epsilon + |1 - \mathbb{P}(X_n \in \mathcal{X}_m)| \\
\leq& 4\epsilon.
\end{aligned}$$

For the same reasons,

$$|\mathbb{E}(g \circ f(X)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X))| \leq 2\epsilon.$$

Let $n_1 \geq n_0$ such that for $n \geq n_1$, $\|f_{n|\mathcal{X}_m} - f_{|\mathcal{X}_m}\| \leq \epsilon$. Since g is $1-$Lipschitz, for $n \geq n_1$,

$$|\mathbb{E}(g \circ f_n(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X_n)|\mathcal{X}_m)| \leq \epsilon.$$

Since $\mathbb{P}(X \in \partial \mathcal{X}_m) = 0$, $(X_n|\mathcal{X}_m)$ converges in law to $(X|\mathcal{X}_m)$ and thus there exists $n_2 \geq n_1$ such that for $n \geq n_2$,

$$|\mathbb{E}(g \circ f(X_n)|\mathcal{X}_m) - \mathbb{E}(g \circ f(X)|\mathcal{X}_m)| \leq \epsilon.$$

Therefore for $n \geq n_2$,

$$|\mathbb{E}(g \circ f_n(X_n)) - \mathbb{E}(g \circ f(X))| \leq 5\epsilon,$$

which implies the Lemma. □

9.6. **Proof of Proposisition 3.** Let us recall here the statement of Proposition 3:

Proposition. *Let U_n be a sequence of $\mathcal{U}^{(2)}$ and $((X^n(i))_{i\geq 1})_{n\geq 1}$ a sequence of random infinite vectors on $[0,1]$. Let $(X^0(1), \dots, X^0(n), \dots)$ be a random infinite vector on $[0,1]$. Suppose that each finite dimensional marginal law of any of these random vectors admits a density with respect to the Lebesgue measure. If $U_n \to U \in \mathcal{U}^{(2)}$ and for each $k \geq 1$, $X_k^n = (X^n(1), \dots, X^n(k))$ converges in law to $X_k^0 = (X^0(1), \dots, X^0(k))$, then for each $k \geq 1$,*

$$\sigma_{U_n}(X_k^n) \longrightarrow_{law} \sigma_U(X_k^0).$$

Proof. Let $k \geq 1$ and set $X = X_k^0$, $X_n = X_k^n$.
Let $A = \bigcup_{\substack{\sigma \in \mathfrak{S}_k \\ f \in C_2^k(A_U)}} X_{f,\sigma}$ (refer to Lemma 8 for the definition of $X_{f,\sigma}$) with the induced topology from $[0,1]^k$, and $B = \mathfrak{S}_k$ with the discrete topology. Then from Lemma 8, $\sigma_U : A \to B$ is constant on each connected component $X_{f,\sigma}$ of A, thus σ_U is continuous.
By the definition of the convergence on $\mathcal{U}^{(2)}$, for $\vec{X} = (X_1, \dots, X_k) \in A$, $\sigma_{U_n}(\vec{X})$ converges to $\sigma_U(\vec{X})$.
Since $[0,1]^k \setminus A$ is of Lebesgue measure 0, we can suppose that X_n, X are random variables on A. It remains to build a sequence of measurable sets $\mathcal{X}_m$ that respects the hypothesis of Lemma 9.
Let $m \geq 1$. For $\eta > 0$, define $\Delta_\eta = \bigcup_{1\leq i,j\leq k}\{(x_1, \dots, x_k) \in [0,1]^k, |x_i - x_j| \leq \eta\}$. Then $\partial_{[0,1]^k}\Delta_\eta \subset \bigcup_{1\leq i,j\leq k}\{(x_1, \dots, x_k) \in [0,1]^k, |x_i - x_j| = \eta\}$. Since the latter is of Lebesgue measure 0, $\mathbb{P}(X \in \partial_{[0,1]^k}\Delta_\eta) = 0$. Since Δ_η is decreasing in η and $Leb(\bigcap \Delta_\eta) = 0$, there exists $\eta_1^m > 0$ such that $\mathbb{P}(X \in \Delta_{\eta_1^m}) \leq \frac{1}{m}$.
Denote by $\mathcal{U} = \{U_\alpha =]r_\alpha, s_\alpha[\}_{\alpha \in \mathcal{A}}$ the finite ordered collection of interval components of $U_\downarrow \cup U_\uparrow$ of size larger than η_1^m. $\mathcal{A}$ is a subset of $A(U)$. For $\eta > 0$, let

$$B_\eta = \bigcup_{i,\alpha}\{(x_1, \dots, x_k) \in [0,1]^k, x_i \in]r_\alpha - \eta, r_\alpha + \eta[\cup]s_\alpha - \eta, s_\alpha + \eta[\}.$$

Once again $Leb(\partial_{[0,1]^k}B_\eta) = 0)$, and since $Leb(\bigcap_\eta B_\eta) = 0$, there exists η_2^m such that $\mathbb{P}(X \in B_{\eta_2^m}) \leq \frac{1}{m}$. Let $K_m = B_{\eta_2^m} \cup \Delta_{\eta_1^m}$, $\mathcal{X}_m$ be the set $\{\vec{x} \notin K\}$. Then $\mathbb{P}(X \in \partial\mathcal{X}_m) = 0$ and $\lim\limits_{m\to+\infty} \mathbb{P}(X \in \mathcal{X}_m) = 1$.
Let $\mathcal{X}_m$ be fixed, with associated complementary set $K_m = B_{\eta_2^m} \cup \Delta_{\eta_1^m}$. Set $\eta_m = \inf(\eta_1^m, \eta_2^m)$, and let n_m be such that for $n \geq n_0$, $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m$. Suppose from

now on that $n \geq n_m$. Since $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m \leq \eta_1^m$ the interval components of $U_\downarrow^n$ (resp. $U_\uparrow^n$) of size larger than η_1^m are in order respecting bijection with those of $U_\downarrow$ (resp. $U_\uparrow$). Denote these interval components of U^n by $\mathcal{U}_n = \{U_\alpha^n =]r_\alpha^n, s_\alpha^n[\}_{\alpha \in \mathcal{A}}$, with $\mathcal{A} \subset A(U_n)$. Moreover since $d_{\mathcal{U}^{(2)}}(U_n, U) \leq \eta_m \leq \eta_2^m$, $|r_\alpha^n - r_\alpha| < \eta_2^m$ and $|s_\alpha^n - s_\alpha| < \eta_2^m$.
Since on $\mathcal{X}_m$, $|x_i - x_j| \geq \eta_1$, if $f \in C_2^k(A(U))$ and $S(f) \not\subset \mathcal{A}$, then $X_{\sigma,f}(U) \cap \mathcal{X}_m = \emptyset$. Thus we can consider that $f \in C_2^k(\mathcal{A})$. The same is true for $f \in C_2^k(A(U_n)), S(f) \not\subset \mathcal{A}$ with $(X_{\sigma,f}(U_n) \cap A) \cap \mathcal{X}_m$.
Let $f \in C_2^k(\mathcal{A})$ and suppose that $\vec{x} \in X_{\sigma,f}(U) \cap \mathcal{X}_m$. Let $\alpha \in S_f$ and suppose that $x_i \in U_\alpha =]r_\alpha, s_\alpha[$; since $\vec{x} \in \mathcal{X}_m$, $x_i \in]r_\alpha + \eta_2^m, s_\alpha - \eta_2^m[$. But $|r_\alpha^n - r_\alpha| < \eta_2^m$ and $|s_\alpha^n - s_\alpha| < \eta_2^m$, thus $x_i \in U_\alpha^n$. Conversely is $\alpha \in S_f$ and $x_i \in U_\alpha^n$, for the same reasons $x_i \in U_\alpha$. This shows that $X_{\sigma,f}(U) \cap \mathcal{X}_m = X_{\sigma,f}(U_n) \cap \mathcal{X}_m$. This yields that $\sigma_{U|X_{\sigma,f}(U)} = \sigma_{U_n|X_{\sigma,f}(U_n)}$. Finally we have proven that for $n \geq n_m$, $\sigma_{U_n|\mathcal{X}_m} = \sigma_{U|\mathcal{X}_m}$, which implies obviously the uniform convergence $\sigma_{U_n|\mathcal{X}_m} \to_{n\to\infty} \sigma_{U|\mathcal{X}_m}$.
The application of Lemma 9 concludes the proposition. $\square$